Portfolio Theory - Exercise Set 2

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Martingale Measures and other considerations

2.1 (Parametrization of Martingale Measures [2p])

Fix the time horizon at T and assume the initial σ-algebra $\mathcal{F}_0$ to be trivial. Let $(S_t^{(0)})_{t\in\mathbf{T}}$ be identically equal to 1 and let $Z_t := \log \frac{S_t}{S_{t-1}}$. Suppose that the market that is described by the pair of processes $S^{(0)}, S^{(1)}$ is arbitrage-free. Suppose that $\mathbb{P}$ is such that the $Z_1, Z_2, \ldots, Z_T$ are i.i.d. with a common normal $\mathcal{N}(\mu, \sigma^2)$ distribution.

Give a relation between the parameters μ and σ^2 if $\mathbb{P}$ is a martingale measure. Is it possible that Z_t has a Gamma distribution instead of a normal one if $\mathbb{P}$ is a martingale measure?

2.2 (Sensitivity of option prices [2p])

Consider an arbitrage-free market with one risky asset. Let $S^{(1)}$ be its price process and $S^{(0)}$ the deterministic price process of the riskless asset. Consider a European call option with discounted payoff

$$\widetilde{H} = \frac{(S_T^{(1)} - K)^+}{S_T^{(0)}},$$

for some $K > 0$. Assume that $S_T^{(1)}$ has a density w.r.t. Lebesgue measure under any risk-neutral measure. Let π^* be an arbitrage-free price of the call option under some risk-neutral measure $\mathbb{P}^*$. Obviously π^* depends on K and $S_0^{(1)}$, so we write $\pi^* = \pi^*(K, S_0^{(1)})$. Show that

$$\frac{\partial \pi^*}{\partial S_0^{(1)}} < 1, \quad \frac{\partial \pi^*}{\partial K} = -\left(1 - F^*(K)\right)\frac{1}{S_T^{(0)}},$$

where F^* is the distribution function of $S_T^{(1)}$ under P^*. To show the first assertion you may make additional assumptions, e.g. that $S_T^{(1)}$ is increasing in $S_0^{(1)}$, or even more explicit, $S_T^{(1)} = S_0^{(1)} R_T$, with R_T a positive random variable.

2.3 (Put-call parity in a multi-period model [2p])

Consider an arbitrage-free market model with a single risky asset, $S^{(1)}$, and a riskless bank account, $S_t^{(0)} = (1+r)^t$, for some $r > -1$. Suppose that an arbitrage-free price π_{call} has been fixed for the discounted claim

$$H_{\text{call}} = \frac{(S_T^{(1)} - K)^+}{S_T^{(0)}}$$

of a European call option with strike $K \geq 0$. Then there exists a nonnegative adapted process $X^{(2)}$ with $X_0^{(2)} = \pi_{\text{call}}$ and $X_T^{(2)} = H_{\text{call}}$ such that the extended market model with discounted price process $(\mathbb{1}, \widetilde{S}^{(1)}, \widetilde{X}^{(2)})$ is arbitrage-free.

Show that the discounted European contingent claim

$$H_{\text{put}} = \frac{(K - S_T^{(1)})^+}{S_T^{(0)}}$$

is attainable in the extended model, and that its unique arbitrage-free price is given by

$$\pi_{\text{put}} = \frac{K}{(1+r)^T} - S_0^{(1)} + \pi_{\text{call}}.$$

2.4 (Additional Information in the Market [2p])
Let's examine a complete market model with finite time horizon $T = 2$ and a discounted price process $(X_t)_{t=0,1,2}$, on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,1,2}, \mathbb{P})$. We denote the unique equivalent martingale measure as $\mathbb{Q}$.

Now, we extend this model by introducing two additional states, ω^+ and ω^-, describing further information that will be uncovered at time $t = 2$. We then define:

$$\begin{aligned}
\widehat{\Omega} &:= \Omega \times \{\omega^+, \omega^-\}, \\
\widehat{\mathcal{F}} &:= \sigma(A \times B | A \in \mathcal{F},\, B \in \{\omega^+, \omega^-\}), \\
\widehat{\mathbb{P}}[\{\omega, \omega^\pm\}] &:= \frac{1}{2}\mathbb{P}[\{\omega\}], \quad \omega \in \Omega.
\end{aligned}$$

The unveiling of additional information at $t = 2$ is reflected by the chosen price process $\hat{X}_t(\omega, \omega^\pm) := X_t(\omega)$ and the following filtration:

$$\begin{aligned}
\widehat{\mathcal{F}}_0 &:= \{\emptyset, \widehat{\Omega}\}, \\
\widehat{\mathcal{F}}_1 &:= \sigma(A \times \{\omega^+, \omega^-\} | A \in \mathcal{F}_1), \\
\widehat{\mathcal{F}}_2 &:= \widehat{\mathcal{F}}.
\end{aligned}$$

For $0 < p < 1$, we further specify the probability measure $\hat{\mathbb{P}}^*_p$ on $\widehat{\mathcal{F}}$ via

$$\widehat{\mathbb{P}}^*_p[A \times \{\omega^+\}] := p \cdot \mathbb{Q}[A], \quad \widehat{\mathbb{P}}^*_p[A \times \{\omega^-\}] := (1-p) \cdot \mathbb{Q}[A] \quad A \in \mathcal{F}.$$

Then solve the following exercises:

(a) Verify that each measure $\hat{\mathbb{P}}^*_p$ is an equivalent martingale measure for the extended model.

(b) Argue that the extended model is incomplete and find a nonattainable contingent claim.

(c) Does every equivalent martingale measure belong to the set $\{\widehat{\mathbb{P}}^*_p \colon 0 < p < 1\}$? If yes, give a proof, if not, find a counterexample.

2.5 (Arbitrage-free prices of non-attainable claims [2p])
Read the remainder of the proof of Theorem 2.9 from the point where we introduce the density $\hat{Z}$. Then prove that the measure $\check{\mathbb{P}}$ in the proof of Theorem 2.9 is a martingale measure for the market $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, X)$.

The Standard Model

Consider a frictionless financial market $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, X)$ with two assets represented by $X = (S^{(0)}, S^{(1)})$. The process of the bank account is given as $S^{(0)}_t = S^{(0)}_0 \mathrm{e}^{rt} = S^{(0)}_0(1+\tilde{r})^t$, $t = 0, 1, \ldots, T$, with fixed interest rate $r \in \mathbb{R}$ and $\tilde{r} = \mathrm{e}^r - 1$; and the risky asset's price follows the model:

$$S^{(1)}_t = S^{(1)}_0 \exp(R_t) = S^{(1)}_0 \prod_{j=1}^{t} (1 + \Delta\widetilde{R}_j), \tag{1}$$

where $\widetilde{R}_j = \sum_{n=1}^{j}(\exp(\Delta R_n) - 1)$, $R_0 = 0$, and the *daily logarithmic returns* $\Delta R_1, \Delta R_2, \ldots$ of $(R_t)_{t \in \mathbf{T}}$ (note that $\Delta R_t = \log(S^{(1)}_t / S^{(1)}_{t-1})$ for $t = 1, \ldots, T$) are assumed to be independent and identically distributed.

We call the market $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, X)$ the *standard model*, whenever the daily logarithmic returns ΔR_t for $t = 1, \ldots, T$ are normally distributed with mean μ and variance σ^2. Note that according to Proposition 2.15 in the lecture notes, the standard model can not be complete as the normal distribution is a continuous distribution and hence, the underyling probability space can not be decomposed into a finite number of atoms!

2.6 (European Call Option in the Standard Model [2p])
Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, X)$ be the *standard model* introduced above. Give a solution to the following problems:

(a) Give an estimator for the parameters μ and σ^2 at time T given the historic data $S_0^{(1)}, S_1^{(1)}, \ldots, S_{T-1}^{(1)}$.

(b) Consider an European Call Option with maturity T and strike K, i.e., an European contingent claim with pay-off at time T given by $H := (S_T^{(1)} - K)^+$.

i) Construct a trading strategy φ such that $W_T(\varphi) \geq H$ $\mathbb{P}$-almost surely.

ii) Let $\pi_L(H) := \sup\{W_0 \in \mathbb{R} \colon \exists \text{ self-financing strategy } \varphi \text{ s.t. } W_0(\varphi) = W_0 \text{ and } W_T(\varphi) \geq H\}$. Show that:

$$\pi_L(H) \geq \max(0, S_0^{(1)} - K\mathrm{e}^{-rT}) = (S_0^{(1)} - K\mathrm{e}^{-rT})^+.$$

(c) Note that the two distributions $\mathcal{N}(\mu, \sigma^2)$ and $\mathcal{N}(r - \frac{1}{2}\tilde{\sigma}^2, \tilde{\sigma}^2)$ are equivalent to the Lebesgue measure and thus also equivalent to each other with density denoted by

$$f_{\tilde{\sigma}} := \frac{\mathrm{d}\mathcal{N}(r - \frac{1}{2}\tilde{\sigma}^2, \tilde{\sigma}^2)}{\mathrm{d}\mathcal{N}(\mu, \sigma^2)}.$$

Next, define a probability measure $\mathbb{Q}_{\tilde{\sigma}} \sim \mathbb{P}$ by

$$\frac{\mathrm{d}\mathbb{Q}_{\tilde{\sigma}}}{\mathrm{d}\mathbb{P}} := \prod_{n=1}^{T} f_{\tilde{\sigma}}(\Delta R_n).$$

Relative to the measure $\mathbb{Q}_{\tilde{\sigma}}$, the price process has the same structure as under $\mathbb{P}$, but with different parameters μ and σ^2. Show that for all $\tilde{\sigma} > 0$ the following holds true:

i) $\mathbb{Q}_{\tilde{\sigma}}$ is a probability measure equivalent to $\mathbb{P}$ and the random variables $\Delta R_1, \ldots, \Delta R_N$ are independent under $\mathbb{Q}_{\tilde{\sigma}}$ with law $\mathcal{N}(r - \tilde{\sigma}^2/2, \tilde{\sigma}^2)$;

ii) $\mathbb{Q}_{\tilde{\sigma}}$ is an equivalent martingale measure.

Note that this then constructs a one-parametric family of equivalent martingale measures $\{\mathbb{Q}_{\tilde{\sigma}} \colon \tilde{\sigma} > 0\}$.

(d) Next, compute the option price that is obtained as an expectation under measure $\mathbb{Q}_{\tilde{\sigma}}$, i.e.,

$$\pi_{\mathbb{Q}_{\tilde{\sigma}}} := S_0^{(0)} \mathbb{E}_{\mathbb{Q}_{\tilde{\sigma}}} \left(\frac{(S_T^{(1)} - K)^+}{S_T^{(0)}} \right) = \mathbb{E}_{\mathbb{Q}_{\tilde{\sigma}}} \left((S_0^{(1)} \mathrm{e}^{R_T - rT} - K\mathrm{e}^{-rT})^+ \right).$$

(e) The options price $\pi_{\mathbb{Q}_{\tilde{\sigma}}}$ depends via $\mathbb{Q}_{\tilde{\sigma}}$ on the parameter $\tilde{\sigma}$. Show that if $\tilde{\sigma} \to 0$ then $\pi_{\mathbb{Q}_{\tilde{\sigma}}} \to (S_0^{(1)} - K\mathrm{e}^{-rT})^+$ and if $\sigma \to \infty$ then $\pi_{\mathbb{Q}_{\tilde{\sigma}}} \to S_0^{(1)}$. (Therefore, the price limits in the standard model coincide with the trivial ones. Put differently, absence of arbitrage does not provide much information on European call prices in the standard model.)