Portfolio Theory - Exercise Set 3

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The Cox-Ross-Rubinstein Model

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, (S^{(0)}, S^{(1)}))$ denote a canonicl CRR model which is complete. Solve the following exercises:

3.1 (Forward-start Options [2p])
Let $T_0 \in \{1, \dots, T-1\}$ and $K > 0$. The payoff of *forward starting call option* has the form

$$\left(\frac{S_T^{(1)}}{S_{T_0}^{(1)}} - K \right)^+ .$$

Determine its arbitrage-free price and replicating strategy in the CRR model.

3.2 (One-period CRR [2p])
Lets assume that $T = 1$, i.e., we assume a one-period CRR model. Suppose we want to determine the price at time zero of the derivative $H = S_1^{(1)}$, i.e., the derivative pays off the stock price at time $T = 1$. What is the time-zero price W_0^H given by the risk-neutral pricing formula?

3.3 (Asian Option [2p])
Consider the three-period CRR model in Figure 1 below and take the interest rate $r = 0.25$. What is D, U, $\mathbb{Q}(R_t = U)$ in this case? For $n = 0, 1, 2, 3$ define

$$Y_n = \sum_{k=0}^{n} S_k^{(1)},$$

the sum of the stock prices between times zero and n. Consider an Asian call option, see Example 2.4 in the lecture notes, that expires at time three and has strike $K = 4$, i.e., whose payoff at time $T = 3$ is

$$H^{\text{asian}} = (\frac{1}{4} Y_3 - 4)^+.$$

Let $W_n^{\text{asian}}(s, y)$ denote the price of this option at time n, if $S_n^{(1)} = s$ and $Y_n = y$. In particular, we have $W_3^{\text{asian}}(s, y) = (\frac{1}{4} y - 4)^+$.

(a) Develop an algorithm for computing W_n^{asian} recursively. In particular, write a formula for W_n^{asian} in terms of W_{n+1}^{asian}.

(b) Apply the algorithm developed in (i) to compute $W_0^{\text{asian}}(4, 4)$, the price of the Asian option at time zero.

(c) Provide a formula for $\delta_n(s, y)$, the number of shares of stock that should be held by the replicating portfolio at time n if $S_n^{(1)} = s$ and $Y_n = y$.

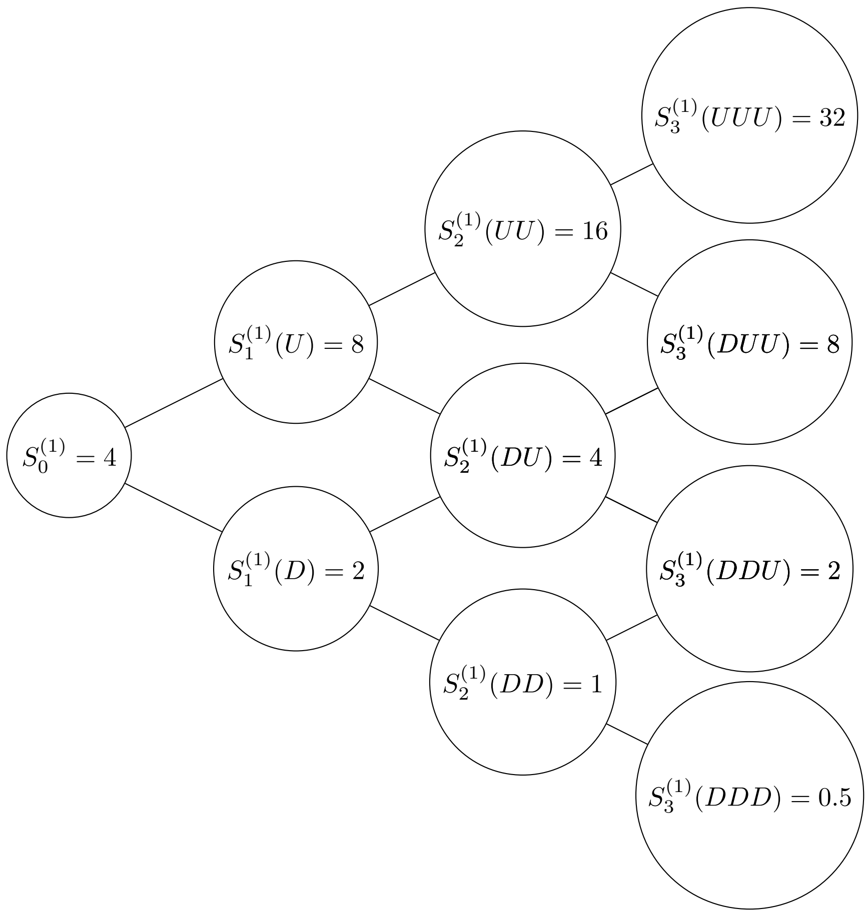

Figure 1: Three-period binomial asset pricing model.

Variance-Optimal Hedging

3.4 (Variance-optimal hedge under martingale measure [4p])
Let X denote the discounted asset price process. In the first exercise we fill in the open gaps in the lecture notes. Indeed, solve the following:

i) Prove the remaining part of Lemma 3.7, i.e., show that
   a) the process $(M_t X_t)_{t\in\mathbf{T}}$ is a martingale;
   b) the Kunita-Watanabe decomposition in equation (46) is unique.

ii) Prove Theorem 3.8.

(Hint: It might help to use the predictable quadratic (co)variation process $\langle M\rangle$ for square-integrable martingales M given by $\Delta\langle M\rangle = \mathbb{E}_{\mathbb{P}}\left[(M_n - M_{n-1})^2|\mathcal{F}_{n-1}\right]$ and its properties for the derivations (see Section 9 in the MTP lecture notes).

3.5 (Variance-optimal hedge in an affine GARCH model [4p])
In this exercise we consider a univariate discrete-time stochastic volatility model of GARCH type given as follows: we model the discounted underlying asset price process $(\widetilde{S}_t)_{t\in\mathbf{T}}$ as

$$\widetilde{S}_t = \widetilde{S}_{t-1}\exp\left(-\frac{1}{2}V_t + \sqrt{V_t}z_t^*\right), \tag{1}$$

$$V_t = \omega + \beta V_{t-1} + \alpha\left(z_{t-1}^* - \gamma^*\sqrt{V_{t-1}}\right)^2, \tag{2}$$

for some suitable parameters ω, α, β and γ such that $V_t \geq 0$ for all $t \in \mathbf{T}$ and where z_t^* is standard normal distributed. The process $(V_t)_{t\in\mathbf{T}}$ is called the *instantaneous variance process* of (the log price of) $\widetilde{S}$. We also assume that the discounted asset price process $(\widetilde{S}_t)_{t\in\mathbf{T}}$ is square-integrable with positive conditional variance process $(\sigma_t^2)_{t=1,2,\dots,T}$ and we denote by $\widetilde{H}$ some discounted square-integrable contingent claim.

i) Argue why a variance-optimal strategy (W_0^*, ϕ^*) for $\widetilde{H}$ exists and provide an expression of the strategy using Theorem 3.8.

ii) Under the additional assumption that $H = f(\widetilde{S}_T)$ for some function f, we have a integral representation for $f\colon \mathbb{C}\to\mathbb{C}$ of the form

$$f(x) = \int_{R-\mathrm{i}\infty}^{R+\mathrm{i}\infty} x^u l(u)\,\mathrm{d}u,$$

for some function l and $R\in\mathbb{R}$. For instance, the payoff of an European Call Option can be written as

$$f(x) = (x-K)^+ = \frac{1}{2\pi\mathrm{i}}\int_{R-\mathrm{i}\infty}^{R+\mathrm{i}\infty} x^u \frac{K^{1-u}}{u(u-1)}\,\mathrm{d}u.$$

   a) Assume that $\widetilde{H}$ has an integral representation as above. Then show that the derivative prices $\widetilde{W}_t^H$ for $t = 0, 1, \dots, T-1$ under some pricing measure $\mathbb{Q}$ and the variance-optimal hedge under the same measure can be expressed using such complex integrals as well.

   b) Take as a fact that the model (1)-(2) is *affine*, which means that for any $t \leq T$ and $T \in \mathbf{T}$ the joint moment-generating function $g(t,T,u,v)$ of $(\widetilde{S}_t, V_t)$ has the following exponential affine form:

$$g(t,T,u,v) = \mathbb{E}_{\mathbb{Q}}\left[\widetilde{S}_T^u \exp\left(vV_{t+2}\right)|\mathcal{F}_t\right] = \widetilde{S}_t^u\exp(A(t,T,u,v) + B(t,T,u,v)V_{t+1}),$$

   for two deterministic functions A and B solving some associated difference equations. Use the representation in a) and this fact to show that

$$\widetilde{W}_t^H = \int_{R-\mathrm{i}\infty}^{R+\mathrm{i}\infty} g(t,T,u,0)l(u)\,\mathrm{d}u,$$

   and that the variance-optimal hedge is given by

$$\phi_{t+1}^* = \int_{R-\mathrm{i}\infty}^{R+\mathrm{i}\infty} \frac{\exp(A(t+1,T,u,0))g(t,t+1,u+1,B(t+1,T,u,0)) - \widetilde{S}_t g(t,T,u,0)}{g(t,t+1,2,0)-\widetilde{S}_t^2} l(u)\,\mathrm{d}u \mathbb{1}_{\{g(t,t+1,2,0)-\widetilde{S}_t^2>0\}}.$$

In the next exercise we construct an example of a financial market, where the bounded mean-variance trade-off condition (43) in the lecture notes is not satisfied and where the subspace $\mathcal{G}_T$ is indeed *not* closed.

3.6 (Counterexample for closedness of the space $\mathcal{G}_T$ [2p])
Let $\Omega = [0,1] \times \{-1,+1\}$ with its Borel σ-algebra $\mathcal{F}$. Outcomes are denoted by $\omega = (u,v)$ with $u \in [0,1], v \in \{-1,+1\}$, and we define $U(\omega) = u$ the first and by $V(\omega) = v$ the second coordinate. Let $\mathcal{F}_0 = \mathcal{F}_1 = \sigma(U)$, $\mathcal{F}_2 = \mathcal{F}$ and let $\mathbb{P}$ be the measure on $(\Omega, \mathcal{F})$ such that U is distributed uniformly on $[0,1]$ and the conditional distribution of V given U is $U^2\delta_{\{+1\}} + (1-U^2)\delta_{\{-1\}}$. Let $X_0 = 0, \Delta X_1 = 1$ and

$$\Delta X_2 = V^+(1+U) - 1 = V^+U - V^-,$$

so that

$$\Delta X_2(u,v) = u\delta_{\{v=+1\}} - \delta_{\{v=-1\}}$$

. Consider now the contingent claim $H = \frac{1}{U}V^+(1+U)$.

i) Show that $H \in L^2(\Omega, \mathcal{F}, \mathbb{P})$

ii) Let ϕ be a predictable process with terminal gain satisfying $G_2(\phi) = H$ $\mathbb{P}$-almost surely. Show that

$$\frac{1}{U}V^+(1+U) = H = \phi_1 \Delta X_1 + \phi_2 \Delta X_2 = \phi_1 + \phi_2(V^+(1+U) - 1) \tag{3}$$

implies that $\phi_1 = \phi_2 = U^{-1}$ $\mathbb{P}$-almost surely.

iii) Show that ϕ is not in $\mathcal{S}^2$ and that therefore H is not in $\mathcal{G}_2$.

iv) Next, set

$$\phi^n := \phi \mathbb{1}_{\{U \geq 1/n\}} = U^{-1}\mathbb{1}_{\{U \geq 1/n\}} \tag{4}$$

and show that $\phi^n \in \mathcal{S}^2$ for every $n \in \mathbb{N}$
and that

$$G_2(\phi^n) = U^{-1}V^+(1+U)\mathbb{1}_{\{U \geq 1/n\}} = H\mathbb{1}_{\{U \geq 1/n\}} \tag{5}$$

converges to H in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

v) Part i)-iv) shows that the space $\mathcal{G}_2$ is not closed in $L^2(\Omega, \mathcal{F}, \mathbb{P})$, so the variance optimization problem for H does not have a solution. To conclude this example, show that X as constructed above does not satisfy the bounded mean-variance trade-off condition.

The Semi-Static Variance-Optimal Hedging Problem

Consider the following extension of variance-optimal hedging, called *semi-static* variance-optimal hedging. The idea is, that in addition to the contingent claim H^0 which is to be hedged, we denote by $H = (H^1, \ldots, H^n)^\top$ the vector of supplementary contingent claims, all assumed to be square-integrable random variables in $L^2(\Omega, \mathcal{F}_T, \mathbb{Q})$. Again, we associate to each H^i the martingale

$$H_t^i := \mathbb{E}\left[H^i \middle| \mathcal{F}_t\right], \qquad t = 0, 1, \ldots, T, \quad i = 0, \ldots, n. \tag{6}$$

The static part of the strategy can be represented by an element v of $\mathbb{R}^n$, where v_i represents the quantity of claim H^i bought at time $t = 0$ and held until time $t = T$. The dynamic part ϑ of the strategy is again represented by an element of $\mathcal{S}^2$, the space of square-integrable predictable processes with respect to the price process S.

The variance-optimal semi-static hedge $(\vartheta, v) \in \mathcal{S}^2 \times \mathbb{R}^n$ and the optimal initial capital $c \in \mathbb{R}$ are the solution of the minimization problem

$$\varepsilon^2 = \min_{(\vartheta,v) \in \mathcal{S}^2 \times \mathbb{R}^n,\, c \in \mathbb{R}} \mathbb{E}\left[\left(c - v^\top \mathbb{E}\left[H_T\right] + \sum_{t=1}^T \vartheta_t \Delta S_t - \left(H_T^0 - v^\top H_T\right)\right)^2\right]. \tag{7}$$

Note that $v^\top \mathbb{E}\left[H_T\right]$ is the cost of setting up the static part of the hedge, and its terminal value is $v^\top H_T$. The dynamic part is self-financing and results in the terminal value $\sum_{t=1}^T \vartheta_t \Delta S_t$. Adding the initial capital c and subtracting the target claim H_T^0 yields the above expression for the hedging problem.

To solve the variance-optimal semi-static hedging problem, we decompose it into an inner and an outer minimization problem and rewrite (7) as

$$\begin{cases} \epsilon^2(v) = \min_{\vartheta\in\mathcal{S}^2,\, c\in\mathbb{R}} \mathbb{E}\left[\left(c - v^\top \mathbb{E}\left[H_T\right] + \sum_{t=1}^{T} \vartheta_t \Delta S_t - \left(H_T^0 - v^\top H_T\right)\right)^2\right], & \text{(inner problem)} \\ \epsilon^2 = \min_{v\in\mathbb{R}^n} \epsilon^2(v). & \text{(outer problem)} \end{cases} \tag{8}$$

The inner problem is of the same form as the variance-optimal hedging problem without supplementary assets, while the outer problem turns out to be a finite-dimensional quadratic optimization problem. To formulate the solution, we write the Kunita-Watanabe decompositions of the claims $(H^0, \ldots, H^n)$ with respect to S as

$$H_t^i = H_0^i + \sum_{s=1}^{t} \vartheta_s^i \Delta S_s + L_t^i, \quad t = 0, 1, \ldots, T, \quad i = 0, \ldots, n. \tag{9}$$

As in the classical variance optimal hedging problem, we obtain the solution:

$$\vartheta_t^i = \frac{\mathbb{E}\left[\Delta H_t^i \Delta S_t \middle|\, \mathcal{F}_{t-1}\right]}{\mathbb{E}\left[(\Delta S_t)^2 \middle|\, \mathcal{F}_{t-1}\right]} \mathbb{1}_{\{\mathbb{E}[(\Delta S_t)^2 |\, \mathcal{F}_{t-1}] \neq 0\}}, \quad t = 1, \ldots, T, \quad i = 0, \ldots, n. \tag{10}$$

We introduce the vector notation $\vartheta = (\vartheta^1, \ldots, \vartheta^n)^\top$ for the strategies and $L = (L^1, \ldots, L^n)^\top$ for the residuals in the Kunita-Watanabe decomposition.

3.7 (Semi-Static Variance-Optimal Hedging [2p])
Consider the variance-optimal semi-static hedging problem (7) and set

$$A := \mathrm{Var}\left[L_T^0\right], \qquad B := \mathrm{Cov}\left[L_T, L_T^0\right], \qquad C := \mathrm{Cov}\left[L_T, L_T\right]. \tag{11}$$

Assume that C is invertible. Show that the unique solution of the semi-static hedging problem is given by

$$c = \mathbb{E}\left[H_T^0\right], \qquad v = C^{-1}B, \qquad \vartheta_t^v = \vartheta_t^0 - v^\top \vartheta_t, \quad t = 1, \ldots, T,$$

and that the minimal squared hedging error is given by

$$\epsilon^2 = A - B^\top C^{-1} B.$$

Moreover, show that the elements of A, B, and C can be expressed as

$$\mathbb{E}\left[L_T^i L_T^j\right] = \mathbb{E}\left[\sum_{t=1}^{T} \mathrm{Cov}\left(\Delta H_t^i, \Delta H_t^j \mid \mathcal{F}_{t-1}\right) - \sum_{t=1}^{T} \vartheta_t^i \vartheta_t^j \mathrm{Var}\left(\Delta S_t \mid \mathcal{F}_{t-1}\right)\right], \quad i, j = 0, \ldots, n. \tag{12}$$