Risk Measures

4.1 (Axiomatic description of risk [3p])
Let ρ be a normalized risk measure on $\mathcal{X}$. Show that any two of the following properties imply the remaining third:

i) Convexity, i.e. $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$ for $\lambda \in [0,1]$ and $X, Y \in \mathcal{X}$;
ii) Positive homogeneity, i.e. $\rho(\lambda X) = \lambda\rho(X)$ for all $\lambda \geq 0$ and $X \in \mathcal{X}$;
iii) Subadditivity, i.e. $\rho(X+Y) \leq \rho(X) + \rho)(Y)$ for $X, Y \in \mathcal{X}$.

4.2 (Valute-at-Risk [3p])
Consider the Valute-at-Risk risk measure VaR_λ from Example 4.9. This is not a convex risk measure, but it can be transformed to one as follows:

We define the **Average Value at Risk** at level $\lambda \in (0,1]$ of a financial position $X \in \mathcal{X}$ as

$$\mathrm{AVaR}_\lambda(X) \coloneqq \frac{1}{\lambda}\int_0^\lambda \mathrm{VaR}_\alpha(X)\,d\alpha.$$

The average value at risk $\mathrm{AVaR}_\lambda(X)$ has the following representation, that you don't have to prove:

$$\mathrm{AVaR}_\lambda(X) = \frac{1}{\lambda}\mathbb{E}_\mathbb{P}\left[(q-X)^+\right] - q = \frac{1}{\lambda}\inf_{r\in\mathbb{R}}\left(\mathbb{E}_\mathbb{P}\left[(r-X)^+\right] - \lambda r\right),$$

for any λ-quantile q of X. Use this and Exercise 4.1 to show that the average value at risk is a coherent risk measure on $\mathcal{X}$.

4.3 (Gini's mean difference [3p])
For any $X \in L_1$ define $\Delta(X) := \mathbb{E}[|X - \tilde{X}|]$, where $\tilde{X}$ is an independent copy of X. The function $\Delta(\cdot)$ is often called **Gini's mean difference** and satisfies $\Delta(X) \in [0, 2\mathbb{E}[|X|]]$, with $\Delta(X) = 0$ if and only if X is P-a.s. constant. In analogy to the mean-standard deviation risk measure in Example 4.13, consider the functional on L_1 defined by

$$\rho_\lambda(X) := \mathbb{E}[-X] + \lambda\Delta(X), \quad \text{for some } \lambda \geq 0.$$

(a) Show that $\rho_\lambda(X)$ is a coherent risk measure for any $\lambda \in [0, \frac{1}{2}]$. Note that a non-constant position X is acceptable if and only if $\mathbb{E}[X] > 0$ and the Gini coefficient of X, defined as

$$G(X) := \frac{\Delta(X)}{2\mathbb{E}[X]},$$

satisfies $G(X) \geq (2\lambda)^{-1}$.

(b) Show that for $\lambda > \frac{1}{2}$ there are nonnegative random variables X such that $\rho_\lambda(X) > 0$. In particular, ρ_λ cannot be monotone for $\lambda > \frac{1}{2}$.

(c) Show that

$$\Delta(X) = 2\int_{-\infty}^{\infty} F(x)(1-F(x))\,dx,$$

where F denotes the distribution function of X, and that

$$\Delta(X) = 4\,\mathrm{cov}(X, F(X))$$

whenever X has a continuous distribution.

(d) Use the above equation to show that $\Delta(X) \leq \frac{2\sigma(X)}{\sqrt{3}}$ for all $X \in L_2$ and conclude that ρ_λ is dominated by the mean-standard deviation risk measure $\rho_{RC}(X) = -\mathbb{E}[X] + c\sigma(X)$ if $\lambda \leq \frac{c\sqrt{3}}{2}$. As in Example 4.9, we denote here by $\sigma(X)$ the square root of the variance of X.

(e) Compute $\Delta(X)$ and $G(X)$ if X has a Pareto distribution with shape parameter $\alpha > 1$ and minimum 1, that is, $\log X$ is exponentially distributed with parameter α.

(f) Consider a log-normally distributed random variable $X = \exp(m + \sigma Z)$, where Z has a standard normal law $N(0,1)$. Show that $G(X) = \mathrm{erf}(\frac{\sigma}{2})$, where $\mathrm{erf}(\cdot)$ is the Gaussian error function, that is,

$$\mathrm{erf}(z) := \mathbb{P}[|Z| \leq z\sqrt{2}] = 2\Phi(z\sqrt{2}) - 1.$$