Portfolio Theory - Exercise Set 5

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Exercises: Expected Utility

5.1 (An utility inequality [3p])
Show that for a utility function $u \in C^1(\mathbb{R})$ it holds that

$$m(\mu) > c(\mu) > \frac{\mathbb{E}\left[Xu'(X)\right]}{\mathbb{E}\left[u'(X)\right]},$$

where X has nondegenerate distribution μ and all expectations are assumed to be finite.

5.2 (CARA utility function [3p])
Let $u(x) = 1 - \exp(-x)$, a CARA function. Consider an investor with utility function u who wants to invest an initial capital. There is one riskless asset, having value 1 and interest rate $r = 0$, and one risky assets with random pay-off S_1 having a normal $\mathcal{N}(m, \sigma^2)$ distribution with $\sigma^2 > 0$. Suppose she invests a fraction λ in the riskless asset and the remainder in the risky asset. The pay-off of this portfolio is thus $\lambda + (1 - \lambda)S_1$. The aim is to maximize her expected utility.

(a) Show that $\mathbb{E}\left[\exp(uS_1)\right] = \exp(um + \frac{1}{2}u^2\sigma^2)$ $(u \in \mathbb{R})$.

(b) Compute for each λ the certainty equivalent of the portfolio.

(c) Let λ^* be the optimal value of λ. Give, by direct computations, conditions on the parameters such that each of the cases $\lambda^* = 0$, $\lambda^* = 1$ or $\lambda^* \in (0, 1)$ occurs.

(d) Compare the results of (c) with the assertions of Proposition 5.13.

5.3 (Decreasing risk aversion [3p])
A utility function $u : \mathbb{R} \to \mathbb{R}$ is said to exhibit *decreasing risk aversion* if the function $x \mapsto \alpha(x)$ (the Arrow-Pratt coefficient) is decreasing. Show that this property is equivalent to saying that for every $x_1 < x_2$ there exists a concave function g such that $u(x_1 + z) = g(u(x_2 + z))$ for all z (for which the given expressions make sense).

5.4 (CARA Again [3p])
Consider the CARA utility function $u(x) = 1 - \exp(-\alpha x)$, $x \in \mathbb{R}$, with $\alpha > 0$, the constant Arrow-Pratt coefficient.

(a) Show that the condition $\mathbb{E}\left[|u(\varphi \cdot Y)|\right] < \infty$ for $\varphi \in \Xi$ of Theorem 6.4 is equivalent to $\mathbb{E}\left[\exp(\varphi \cdot Y)\right] < \infty$, for all $\varphi \in \mathbb{R}^d$.

(b) Show that the risk-neutral measure $\mathbb{P}^*$ of Proposition 6.5 is the same for all $\alpha > 0$.

(c) Suppose that Y has a d-dimensional multivariate normal distribution with mean vector m and invertible covariance matrix Σ. Compute the optimal $\varphi^* \in \mathbb{R}^d$.

5.5 (Normal and inferior goods [3p])
Given an arbitrage free market and a utility function $\tilde{u}$ as at the beginning of this section, the transformed utility function u depends on the initial capital W_0. In general, an optimal portfolio will also depend on W_0. We study this for the case $d = 1$ and zero interest, i.e. $r = 0$. Assume that $\tilde{u}$ is a C^2 function and that everywhere below interchanging of expectation and differentiation is allowed. Put

$$f(w, \varphi) = \mathbb{E}\left[\tilde{u}'(W_0 + \varphi Y)Y\right]$$

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(a) Show that $\frac{\partial f}{\partial \varphi}(W_0, \varphi) < 0$.

(b) Conclude that locally for every $W_0 > 0$, there is a continuously differentiable function $x \mapsto \varphi^*(x)$ such that $f(W_0, \varphi^*(W_0)) = 0$.

(c) Show that

$$\frac{\mathrm{d}\varphi^*(W_0)}{\mathrm{d}W_0} = -\frac{\mathbb{E}\left[\tilde{u}''(W_0 + \varphi^*(W_0)Y)Y\right]}{\mathbb{E}\left[\tilde{u}''(\varphi^*(W_0)Y + W_0)Y^2\right]}$$

(d) Assume that $\mathbb{E}[Y] > 0$ and that Arrow-Pratt coefficient $\tilde{\alpha}(\cdot)$ of $\tilde{u}$ is a decreasing function. Show that $Y\tilde{\alpha}(W_0 + \varphi^*(W_0)Y) \leq Y\tilde{\alpha}(W_0)$.

(e) Conclude, under the assumptions in (d), that $\varphi^*(\cdot)$ is an increasing function of W_0. (In Microeconomics, assets with the latter property are called *normal goods*. Assets with decreasing demand ξ^* are called *inferior goods*.)

5.6 (Relative Entropy and Poisson distribution [3p])
Consider a market with one risky good, its value at $t = 1$ is S_1 and price S_0 (at $t = 0$). Assume that S_1 has under $\mathbb{P}$ a Poisson distribution with parameter $\alpha > 0$. Consider the family of probability measures $\mathbb{P}_\lambda$ on $(\Omega, \mathcal{F})$ with $\lambda \in \mathbb{R}_d$ given by

$$\frac{\mathrm{d}\mathbb{P}_\lambda}{\mathrm{d}\mathbb{P}} = \frac{\mathrm{e}^{\lambda \cdot Y}}{\mathbb{E}\left[\mathrm{e}^{\lambda \cdot Y}\right]},$$

where Y is the discounted net gains (see Section 6).

(a) Show that $\mathbb{E}\left[\mathrm{e}^{\lambda \cdot Y}\right] < \infty$ for all $\lambda \in \mathbb{R}$

(b) Show that S_1 has a Poisson distribution with parameter $\alpha \mathrm{e}^\lambda$ under $\mathbb{P}_\lambda$.

(c) Compute the minimizer of $\lambda \mapsto \mathbb{E}\left[\mathrm{e}^{\lambda \cdot Y}\right]$ directly.

(d) Show that the minimizer λ^* satisfies $\mathbb{E}_{\mathbb{P}_{\lambda^*}}[Y] = 0$.