Portfolio Theory - Exercise Set 6
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Exercises: Optimal Control

6.1 (Maximization of expected utility from terminal wealth in the CRR model [3p])
Consider a CRR model, in which the returns are *iid* with

$$P(R_t = U) = p \quad \text{(where } p \text{ is not necessarily equal to the risk neutral value } p^*\text{)}.$$

Consider the maximization of the expected utility of terminal wealth $\mathbb{E}[u(W_T)]$, with $u(x) = \log x$. Compute the optimal trading strategy φ^* via dynamic programming.

(Hint: show that $\hat{v}_t(x) = \log x + k_t$ for some constants k_t and $\hat{v}_t$ as in equation (96) of the lecture notes.)

6.2 (Optimality Principle [4p])
Prove the "Optimality Principle" Proposition 8.8.

6.3 (Logarithmic utility of terminal wealth [4p])
Assume we are in the same setting of Section 8.3.1 with $d = 1$ and $u(x) = \log(x)$, but with $(R_t)_{t\in\mathbf{T}}$ not necessarily *iid*. Show using Proposition 8.9 that the optimal control $\alpha^*(t, \omega)$ in this situation is given as the maximizer of the mapping $\gamma \mapsto \mathbb{E}_{\mathbb{P}}\left[\log(1 + \gamma R_t) \mid \mathcal{F}_{t-1}\right](\omega)$.

6.4 (Optimal control variance-optimal hedging [3p])
Let us unsider the variance-optimal hedging problem, where $(S_t)_{t\in\mathbf{T}}$ denotes a one-dimensional square-integrable asset price process and $H \in \mathcal{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$ an European contingent claim. Moreover, we assume that $(S_t)_{t\in\mathbf{T}}$ is already a martingale under $\mathbb{P}$. We already know from Theorem 3.8 that the variance-optimal hedge (W_0^*, ϕ^*) is given by:

$$\begin{cases} \phi^*_{t+1} & := \frac{\operatorname{Cov}(\Delta \widehat{W}_{t+1}, \Delta X_{t+1} | \mathcal{F}_t)}{\sigma^2_{t+1}} \mathbb{1}_{\{\sigma_{t+1} \neq 0\}}, \quad t = 0, \dots, T-1, \\ W_0^* & := \widehat{W}_0 = \mathbb{E}_{\mathbb{P}}\left[\widetilde{H}\right]. \end{cases}$$

Identify the variance-optimal hedging problem with a stochastic optimal control problem and show that the variance-optimal hedge (W_0^*, ϕ^*) solves the optimal control problem using Proposition 8.9 in the lecture notes.

6.5 (Maximizing expected utility from consumption [4p])
Consider the maximization of $\mathbb{E}_{\mathbb{P}}\left[\sum_{t=0}^{T} \beta^t u(C_t)\right]$ in the setting of Section 8.4.3, with the power utility function $u(x) = x^\gamma/\gamma$ for $\gamma < 1$ and $\gamma \neq 0$. Show that the optimal C_t^* is of the form

$$C_t^* = c \frac{\beta^{t/(1-\gamma)}}{N_t^{1/(1-\gamma)}},$$

for some c (which one?) and compute the maximal value of the objective function.