Portfolio Theory - Exercise Set 3

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Variance-Optimal Hedging

3.1 (Variance-optimal hedge in an affine GARCH model [4p])
In this exercise we consider a univariate discrete-time stochastic volatility model of GARCH type given as follows: we model the discounted underlying asset price process $(\widetilde{S}_t)_{t\in\mathbf{T}}$ as

$$\widetilde{S}_t = \widetilde{S}_{t-1} \exp\big(-\frac{1}{2}V_t + \sqrt{V_t} z_t^*\big), \tag{1}$$

$$V_t = \omega + \beta V_{t-1} + \alpha\big(z_{t-1}^* - \gamma^* \sqrt{V_{t-1}}\big)^2, \tag{2}$$

for some suitable parameters ω, α, β and γ such that $V_t \geq 0$ for all $t \in \mathbf{T}$ and where z_t^* is standard normal distributed. The process $(V_t)_{t\in\mathbf{T}}$ is called the *instantaneous variance process* of (the log price of) $\widetilde{S}$. We also assume that the discounted asset price process $(\widetilde{S}_t)_{t\in\mathbf{T}}$ is square-integrable with positive conditional variance process $(\sigma_t^2)_{t=1,2,\dots,T}$ and we denote by $\widetilde{H}$ some discounted square-integrable contingent claim.

i) Argue why a variance-optimal strategy (W_0^*, ϕ^*) for $\widetilde{H}$ exists and provide an expression of the strategy using Theorem 3.8.

ii) Under the additional assumption that $H = f(\widetilde{S}_T)$ for some function f, we have a integral representation for $f\colon \mathbb{C} \to \mathbb{C}$ of the form

$$f(x) = \int_{R-\mathrm{i}\,\infty}^{R+\mathrm{i}\,\infty} x^u l(u)\,\mathrm{d}u,$$

for some function l and $R \in \mathbb{R}$. For instance, the payoff of an European Call Option can be written as

$$f(x) = (x-K)^+ = \frac{1}{2\pi\mathrm{i}} \int_{R-\mathrm{i}\,\infty}^{R+\mathrm{i}\,\infty} x^u \frac{K^{1-u}}{u(u-1)}\,\mathrm{d}u.$$

a) Assume that $\widetilde{H}$ has an integral representation as above. Then show that the derivative prices $\widetilde{W}_t^H$ for $t = 0, 1, \dots, T-1$ under some pricing measure $\mathbb{Q}$ and the variance-optimal hedge under the same measure can be expressed using such complex integrals as well.

b) Take as a fact that the model (1)-(2) is *affine*, which means that for any $t \leq T$ and $T \in \mathbf{T}$ the joint moment-generating function $g(t, T, u, v)$ of $(\widetilde{S}_t, V_t)$ has the following exponential affine form:

$$g(t,T,u,v) = \mathbb{E}_{\mathbb{Q}}\left[\widetilde{S}_T^u \exp\left(vV_{t+2}\right) | \mathcal{F}_t\right] = \widetilde{S}_t^u \exp(A(t,T,u,v) + B(t,T,u,v)V_{t+1}),$$

for two deterministic functions A and B solving some associated difference equations. Use the representation in a) and this fact to show that

$$\widetilde{W}_t^H = \int_{R-\mathrm{i}\,\infty}^{R+\mathrm{i}\,\infty} g(t,T,u,0) l(u)\,\mathrm{d}u,$$

and that the variance-optimal hedge is given by

$$\phi_{t+1}^* = \int_{R-\mathrm{i}\,\infty}^{R+\mathrm{i}\,\infty} \frac{\exp(A(t+1,T,u,0)) g(t,t+1,u+1,B(t+1,T,u,0)) - \widetilde{S}_t g(t,T,u,0)}{g(t,t+1,2,0) - \widetilde{S}_t^2} l(u)\,\mathrm{d}u \mathbb{1}_{\{g(t,t+1,2,0)-\widetilde{S}_t^2>0\}}.$$

In the next exercise we construct an example of a financial market, where the bounded mean-variance trade-off condition (43) in the lecture notes is not satisfied and where the subspace $\mathcal{G}_T$ is indeed *not* closed.

3.2 (Counterexample for closedness of the space $\mathcal{G}_T$ [3p])
Let $\Omega = [0,1] \times \{-1,+1\}$ with its Borel σ-algebra $\mathcal{F}$. Outcomes are denoted by $\omega = (u,v)$ with $u \in [0,1], v \in \{-1,+1\}$, and we define $U(\omega) = u$ the first and by $V(\omega) = v$ the second coordinate. Let $\mathcal{F}_0 = \mathcal{F}_1 = \sigma(U)$, $\mathcal{F}_2 = \mathcal{F}$ and let $\mathbb{P}$ be the measure on $(\Omega, \mathcal{F})$ such that U is distributed uniformly on $[0,1]$ and the conditional distribution of V given U is $U^2\delta_{\{+1\}} + (1-U^2)\delta_{\{-1\}}$. Let $X_0 = 0, \Delta X_1 = 1$ and

$$\Delta X_2 = V^+(1+U) - 1 = V^+U - V^-,$$

so that

$$\Delta X_2(u,v) = u\delta_{\{v=+1\}} - \delta_{\{v=-1\}}$$

. Consider now the contingent claim $H = \frac{1}{U}V^+(1+U)$.

i) Show that $H \in L^2(\Omega, \mathcal{F}, \mathbb{P})$
ii) Let ϕ be a predictable process with terminal gain satisfying $G_2(\phi) = H$ $\mathbb{P}$-almost surely. Show that

$$\frac{1}{U}V^+(1+U) = H = \phi_1\Delta X_1 + \phi_2\Delta X_2 = \phi_1 + \phi_2(V^+(1+U) - 1) \tag{3}$$

implies that $\phi_1 = \phi_2 = U^{-1}$ $\mathbb{P}$-almost surely.
iii) Show that ϕ is not in $\mathcal{S}^2$ and that therefore H is not in $\mathcal{G}_2$.
iv) Next, set

$$\phi^n := \phi \mathbb{1}_{\{U \geq 1/n\}} = U^{-1}\mathbb{1}_{\{U \geq 1/n\}} \tag{4}$$

and show that $\phi^n \in \mathcal{S}^2$ for every $n \in \mathbb{N}$
and that

$$G_2(\phi^n) = U^{-1}V^+(1+U)\mathbb{1}_{\{U \geq 1/n\}} = H\mathbb{1}_{\{U \geq 1/n\}} \tag{5}$$

converges to H in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.
v) Part i)-iv) shows that the space $\mathcal{G}_2$ is not closed in $L^2(\Omega, \mathcal{F}, \mathbb{P})$, so the variance optimization problem for H does not have a solution. To conclude this example, show that X as constructed above does not satisfy the bounded mean-variance trade-off condition.

Consider the following extension of variance-optimal hedging, called *semi-static* variance-optimal hedging. The idea is, that in addition to the contingent claim H^0 which is to be hedged, we denote by $H = (H^1, \ldots, H^n)^\top$ the vector of supplementary contingent claims, all assumed to be square-integrable random variables in $L^2(\Omega, \mathcal{F}_T, \mathbb{Q})$. Again, we associate to each H^i the martingale

$$H_t^i := \mathbb{E}\left[H^i \middle| \mathcal{F}_t\right], \qquad t = 0,1,\ldots,T, \quad i = 0,\ldots,n. \tag{6}$$

The static part of the strategy can be represented by an element v of $\mathbb{R}^n$, where v_i represents the quantity of claim H^i bought at time $t = 0$ and held until time $t = T$. The dynamic part ϑ of the strategy is again represented by an element of $\mathcal{S}^2$, the space of square-integrable predictable processes with respect to the price process S.

The variance-optimal semi-static hedge $(\vartheta, v) \in \mathcal{S}^2 \times \mathbb{R}^n$ and the optimal initial capital $c \in \mathbb{R}$ are the solution of the minimization problem

$$\varepsilon^2 = \min_{(\vartheta,v)\in\mathcal{S}^2\times\mathbb{R}^n,\, c\in\mathbb{R}} \mathbb{E}\left[\left(c - v^\top\mathbb{E}\left[H_T\right] + \sum_{t=1}^T \vartheta_t\Delta S_t - \left(H_T^0 - v^\top H_T\right)\right)^2\right]. \tag{7}$$

Note that $v^\top\mathbb{E}\left[H_T\right]$ is the cost of setting up the static part of the hedge, and its terminal value is $v^\top H_T$. The dynamic part is self-financing and results in the terminal value $\sum_{t=1}^T \vartheta_t\Delta S_t$. Adding the initial capital c and subtracting the target claim H_T^0 yields the above expression for the hedging problem.

To solve the variance-optimal semi-static hedging problem, we decompose it into an inner and an outer minimization problem and rewrite (7) as

$$\begin{cases} \epsilon^2(v) = \min_{\vartheta\in\mathcal{S}^2,\, c\in\mathbb{R}} \mathbb{E}\left[\left(c - v^\top\mathbb{E}\left[H_T\right] + \sum_{t=1}^T \vartheta_t\Delta S_t - \left(H_T^0 - v^\top H_T\right)\right)^2\right], & \text{(inner problem)} \\ \epsilon^2 = \min_{v\in\mathbb{R}^n} \epsilon^2(v). & \text{(outer problem)} \end{cases} \tag{8}$$

The inner problem is of the same form as the variance-optimal hedging problem without supplementary assets, while the outer problem turns out to be a finite-dimensional quadratic optimization problem. To formulate the solution, we write the Kunita-Watanabe decompositions of the claims $(H^0, \dots, H^n)$ with respect to S as

$$H^i_t = H^i_0 + \sum_{s=1}^{t} \vartheta^i_s \Delta S_s + L^i_t, \quad t = 0, 1, \dots, T, \quad i = 0, \dots, n. \tag{9}$$

As in the classical variance optimal hedging problem, we obtain the solution:

$$\vartheta^i_t = \frac{\mathbb{E}\left[\Delta H^i_t \Delta S_t \middle| \mathcal{F}_{t-1}\right]}{\mathbb{E}\left[(\Delta S_t)^2 \middle| \mathcal{F}_{t-1}\right]} \mathbb{1}_{\{\mathbb{E}[(\Delta S_t)^2 | \mathcal{F}_{t-1}] \neq 0\}}, \quad t = 1, \dots, T, \quad i = 0, \dots, n. \tag{10}$$

We introduce the vector notation $\vartheta = (\vartheta^1, \dots, \vartheta^n)^\top$ for the strategies and $L = (L^1, \dots, L^n)^\top$ for the residuals in the Kunita-Watanabe decomposition.

3.3 (Semi-Static Variance-Optimal Hedging [3p])
Consider the variance-optimal semi-static hedging problem (7) and set

$$A := \operatorname{Var}\left[L^0_T\right], \qquad B := \operatorname{Cov}\left[L_T, L^0_T\right], \qquad C := \operatorname{Cov}\left[L_T, L_T\right]. \tag{11}$$

Assume that C is invertible. Show that the unique solution of the semi-static hedging problem is given by

$$c = \mathbb{E}\left[H^0_T\right], \qquad v = C^{-1}B, \qquad \vartheta^v_t = \vartheta^0_t - v^\top \vartheta_t, \quad t = 1, \dots, T,$$

and that the minimal squared hedging error is given by

$$\epsilon^2 = A - B^\top C^{-1} B.$$

Moreover, show that the elements of A, B, and C can be expressed as

$$\mathbb{E}\left[L^i_T L^j_T\right] = \mathbb{E}\left[\sum_{t=1}^{T} \operatorname{Cov}\left(\Delta H^i_t, \Delta H^j_t \mid \mathcal{F}_{t-1}\right) - \sum_{t=1}^{T} \vartheta^i_t \vartheta^j_t \operatorname{Var}\left(\Delta S_t \mid \mathcal{F}_{t-1}\right)\right], \quad i, j = 0, \dots, n. \tag{12}$$

Risk Measures

3.4 (Gini's mean difference [4p])
For any $X \in L_1$ define $\Delta(X) := \mathbb{E}[|X - \tilde{X}|]$, where $\tilde{X}$ is an independent copy of X. The function $\Delta(\cdot)$ is often called **Gini's mean difference** and satisfies $\Delta(X) \in [0, 2\mathbb{E}[|X|]]$, with $\Delta(X) = 0$ if and only if X is P-a.s. constant. In analogy to the mean-standard deviation risk measure in Example 4.13, consider the functional on L_1 defined by

$$\rho_\lambda(X) := \mathbb{E}[-X] + \lambda \Delta(X), \quad \text{for some } \lambda \geq 0.$$

(a) Show that $\rho_\lambda(X)$ is a coherent risk measure for any $\lambda \in [0, \frac{1}{2}]$. Note that a non-constant position X is acceptable if and only if $\mathbb{E}[X] > 0$ and the Gini coefficient of X, defined as

$$G(X) := \frac{\Delta(X)}{2\mathbb{E}[X]},$$

satisfies $G(X) \geq (2\lambda)^{-1}$.

(b) Show that for $\lambda > \frac{1}{2}$ there are nonnegative random variables X such that $\rho_\lambda(X) > 0$. In particular, ρ_λ cannot be monotone for $\lambda > \frac{1}{2}$.

(c) Show that

$$\Delta(X) = 2 \int_{-\infty}^{\infty} F(x)(1 - F(x))\, dx,$$

where F denotes the distribution function of X, and that

$$\Delta(X) = 4 \operatorname{cov}(X, F(X))$$

whenever X has a continuous distribution.

(d) Use the above equation to show that $\Delta(X) \leq \frac{2\sigma(X)}{\sqrt{3}}$ for all $X \in L_2$ and conclude that ρ_λ is dominated by the mean-standard deviation risk measure $\rho_{RC}(X) = -\mathbb{E}[X] + c\sigma(X)$ if $\lambda \leq \frac{c\sqrt{3}}{2}$. As in Example 4.9, we denote here by $\sigma(X)$ the square root of the variance of X.

(e) Compute $\Delta(X)$ and $G(X)$ if X has a Pareto distribution with shape parameter $\alpha > 1$ and minimum 1, that is, $\log X$ is exponentially distributed with parameter α.

(f) Consider a log-normally distributed random variable $X = \exp(m + \sigma Z)$, where Z has a standard normal law $N(0,1)$. Show that $G(X) = \operatorname{erf}(\frac{\sigma}{2})$, where $\operatorname{erf}(\cdot)$ is the Gaussian error function, that is,

$$\operatorname{erf}(z) := \mathbb{P}[|Z| \leq z\sqrt{2}] = 2\Phi(z\sqrt{2}) - 1.$$