Portfolio Theory - Exercise Set 4

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Exercises: Some Utility Theory and Portfolio Optimization

4.1 (CARA utility function [3p])
Let $u(x) = 1 - \exp(-x)$, a CARA function. Consider an investor with utility function u who wants to invest an initial capital. There is one riskless asset, having value 1 and interest rate $r = 0$, and one risky assets with random pay-off S_1 having a normal $\mathcal{N}(m, \sigma^2)$ distribution with $\sigma^2 > 0$. Suppose she invests a fraction λ in the riskless asset and the remainder in the risky asset. The pay-off of this portfolio is thus $\lambda + (1 - \lambda)S_1$. The aim is to maximize her expected utility.

(a) Show that $\mathbb{E}[\exp(uS_1)] = \exp(um + \frac{1}{2}u^2\sigma^2)$ $(u \in \mathbb{R})$.

(b) Compute for each λ the certainty equivalent of the portfolio.

(c) Let λ^* be the optimal value of λ. Give, by direct computations, conditions on the parameters such that each of the cases $\lambda^* = 0$, $\lambda^* = 1$ or $\lambda^* \in (0, 1)$ occurs.

(d) Compare the results of iii) with the assertions of Proposition D.13.

4.2 (Maximization of expected utility from terminal wealth in the CRR model [3p])
Consider the CRR model, in which the returns are *iid* with

$$P(R_t = U) = p \quad \text{(where } p \text{ is not necessarily equal to the risk neutral value } p^*\text{)}.$$

Consider the maximization of the expected utility of terminal wealth $\mathbb{E}[u(W_T)]$, with $u(x) = \log x$. Compute the optimal trading strategy φ^* via dynamic programming.

(Hint: show that $\hat{v}_t(x) = \log x + k_t$ for some constants k_t and $\hat{v}_t$ as in equation (95) of the lecture notes.)

4.3 (Optimality Principle [4p])
Prove the "Optimality Principle" Proposition 6.10.

4.4 (Logarithmic utility of terminal wealth [4p])
Assume we are in the same setting of Section 6.3.1 with $d = 1$ and $u(x) = \log(x)$, but with $(R_t)_{t \in \mathbf{T}}$ not necessarily *iid*. Show using Proposition 6.10 that the optimal control $\alpha^*(t, \omega)$ in this situation is given as the maximizer of the mapping $\gamma \mapsto \mathbb{E}_{\mathbb{P}}[\log(1 + \gamma R_t) \mid \mathcal{F}_{t-1}](\omega)$.

4.5 (Optimal control variance-optimal hedging [4p])
Let us unsider the variance-optimal hedging problem, where $(S_t)_{t \in \mathbf{T}}$ denotes a one-dimensional square-integrable asset price process and $H \in \mathcal{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$ an European contingent claim. Moreover, we assume that $(S_t)_{t \in \mathbf{T}}$ is already a martingale under $\mathbb{P}$. We already know from Theorem 3.9 that the variance-optimal hedge (W_0^*, ϕ^*) is given by:

$$\begin{cases} \phi_{t+1}^* & := \frac{\mathrm{Cov}(\Delta \widehat{W}_{t+1}, \Delta X_{t+1} | \mathcal{F}_t)}{\sigma_{t+1}^2} \mathbb{1}_{\{\sigma_{t+1} \neq 0\}}, \quad t = 0, \dots, T-1, \\ W_0^* & := \widehat{W}_0 = \mathbb{E}_{\mathbb{P}}\left[\widetilde{H}\right]. \end{cases}$$

Identify the variance-optimal hedging problem with a stochastic optimal control problem and show that the variance-optimal hedge (W_0^*, ϕ^*) solves the optimal control problem using Proposition 6.10 in the lecture notes.