Portfolio Theory - Exercise Set 5

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Exam-like Questions

5.1 (Fundamentals [2p])
Consider a discrete-time financial market model.

(a) Prove that if there exists an equivalent martingale measure $\mathbb{Q}$, then the market admits no arbitrage opportunities.

(b) Define what it means for the market to be complete. State the Second Fundamental Theorem of Asset Pricing, which links market completeness to the uniqueness of the equivalent martingale measure.

(c) Give an example of a contigent claim which is always attainable.

(d) Now, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,1}, \mathbb{P})$ be a finite filtered probability space with $\Omega = \{\omega_1, \omega_2\}$. Consider a one-period financial market with a riskless bond (price $B_0 = 1$, $B_1 = 1$) and a risky asset with price process $(S_t)_{t=0,1}$ given by:

$$S_0 = 1, \quad S_1(\omega_1) = 2, \quad S_1(\omega_2) = 0.5.$$

Assume that $\mathbb{P}[\omega_1] = \mathbb{P}[\omega_2] = 0.5$.

(i) Define the concept of an equivalent martingale measure (EMM) in this setting.

(ii) Find all probability measures $\mathbb{Q}$ on Ω that are equivalent to $\mathbb{P}$ and under which (S_t) is a $\mathbb{Q}$-martingale.

(iii) Is the EMM unique? Interpret your result in terms of market completeness.

5.2 (Binomial Model [2p])
Consider a one-period market with a risk-free bond and a single risky asset. Let S_0 be the initial stock price. At time 1, the stock price is S_1 which can take two values: $S_1^u = u\,S_0$ (up) or $S_1^d = d\,S_0$ (down), with $0 < d < (1+r) < u$. The bank account is $B_0 = 1$ and $B_1 = 1 + r$.

(a) Determine the risk-neutral probability q for this model (i.e. find q such that the expected gross return of S_1 under q equals $1 + r$).

(b) Let $H = (S_1 - K)^+$ be a European call option paying off H at time 1 (for some strike K). Compute the no-arbitrage price C_0 of this option at time 0 using risk-neutral pricing.

(c) Find a self-financing hedging strategy (specify the holdings in the bond and the stock at time 0) that replicates the payoff H at time 1.

5.3 (Risk Measures [2p])
Let $\rho(X)$ be a risk measure assigned to a financial position (random payoff) X.

(a) Define the following properties of ρ: monotonicity (with respect to X), cash invariance, convexity, and positive homogeneity. Hence, give the definition of a coherent risk measure in terms of these properties.

(b) Show that if ρ is convex and positive homogeneous, then ρ is subadditive.

(c) Is the Value-at-Risk risk measure coherent? Justify your answer (specifically, state which of the coherence axioms VaR satisfies or fails).

5.3 (Stochastic Optimal Control [1p])
Consider a finite-horizon portfolio optimization problem.

(a) Let $V_t(w)$ denote the investor's value function at time t with current wealth w. Write down the Bellman optimality equation (dynamic programming recursion) that $V_t(w)$ satisfies for a portfolio selection problem over $t = 0, 1, \ldots, T$.

(b) Now specialize to a one-period model ($T = 1$). An investor with initial wealth w can invest a fraction α of wealth in a risky asset (stock) and the rest in a risk-free bond (with interest rate r). The stock price will either up-move by a factor $u > 1$ or down-move by $d < 1$ (with given probabilities p and $1 - p$). The investor's utility function is $u(x) = \ln(x)$. Determine the optimal fraction α^* of wealth to invest in the stock at $t = 0$ that maximizes the expected utility $E[\ln W_1]$ of terminal wealth.

5.4 (Variance-Optimal Hedging[1p])

[Variance-Optimal Hedging in a One-Period Market] Consider a one-period incomplete market model with riskless bond $B_0 = B_1 = 1$ and a single risky asset with $S_0 = 1$. At time $t = 1$, the asset price is random with

$$\mathbb{P}[S_1 = 1.5] = 0.6, \qquad \mathbb{P}[S_1 = 0.7] = 0.4.$$

Let H be a contingent claim with payoff

$$H = \begin{cases} 1.2 & \text{if } S_1 = 1.5, \\ 0.4 & \text{if } S_1 = 0.7. \end{cases}$$

(a) Explain why this market is incomplete. What is the variance-optimal hedging problem?

(b) Determine the self-financing strategy (x, y) (cash + risky asset) that minimizes the expected squared hedging error:

$$\mathbb{E}\left[(x + yS_1 - H)^2\right].$$

(c) Compute the minimal variance and interpret the meaning of the optimal strategy.