Portfolio Theory - Exercise Set 1

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Exercises: Modelling Financial Markets

We assume that $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, X)$ is a frictionless financial market in finite discrete-time $\mathbf{T} = \{0, 1, \ldots, T\}$ and $(d+1)$-tradable assets with $d \in \mathbb{N}$. If φ_t and X_t are two $(d+1)$-dimensional vectors we denote the scalar product as $\varphi_t^\intercal X_t$, but you might also write $\varphi_t \cdot X_t$ or $\langle \varphi_t, X_t \rangle_{\mathbb{R}^d}$ if you prefer. For any stochastic process $(Y_t)_{t \in \mathbf{T}}$ we write $\Delta Y_t = Y_t - Y_{t-1}$ for all $t \geq 1$ and set $\Delta Y_0 = Y_0$.

Now, solve the following exercises:

1.1 (Predictable or Adapted? [3p])

Consider a financial market with only two assets which are modeled as usual by the stochastic process $X = (S^{(0)}, S^{(1)})$ and that is adapted to the filtration $(\mathcal{F}_t)_{t \in \mathbf{T}}$.

Decide which of the following processes φ are predictable and which in general are not:

(a) $\varphi_t = \mathbf{1}_{\{S_t^{(1)} > S_{t-1}^{(1)}\}}$;

(b) $\varphi_1 = 1$ and $\varphi_t = \mathbf{1}_{\{S_{t-1}^{(1)} > S_{t-2}^{(1)}\}}$ for $t \geq 2$;

(c) $\varphi_t = \mathbf{1}_A \cdot \mathbf{1}_{\{t > t_0\}}$, where $t_0 \in \{0, \ldots, T\}$ and $A \in \mathcal{F}_{t_0}$;

(d) $\varphi_t = \mathbf{1}_{\{S_t^{(1)} > S_0^{(1)}\}}$;

(e) $\varphi_1 = 1$ and $\varphi_t = 2\varphi_{t-1}\mathbf{1}_{\{S_{t-1}^{(1)} < S_0^{(1)}\}}$ for $t \geq 2$.

1.2 (Self-financing strategies [3p])

Let $\varphi = (\varphi_t)_{t \in \mathbf{T}}$ be a trading strategy such that $\varphi_t^\intercal X_t = \varphi_{t+1}^\intercal X_t$ for all $t \in \{0, 1, \ldots, T-1\}$. Show that this is equivalent to

$$\Delta W_t(\varphi) = \varphi_t^\intercal \Delta X_t, \quad \forall t \in \{1, 2, \ldots, T\},$$

where $(W_t(\varphi))_{t \in \mathbf{T}}$ denotes the wealth process of the strategy φ. Whenever a trading stratefy satisfies one of the equivalent conditions, we call the strategy φ *self-financing*.

1.3 (A geometric model [3p])

Let $(S_t)_{t \in \mathbf{T}}$ be a strictly positive price process of some risky asset, say a stock. The returns are defined by $R_t = \Delta S_t / S_{t-1}$ for $t = 1, 2, \ldots, T$ so that $S_t = S_0 \prod_{k=1}^t (1 + R_k)$ and let the filtration $\mathbb{F}$ be given by $\mathcal{F}_t = \sigma(S_0, \ldots, S_t)$. Then:

i) Show that $(S_t)_{t \in \mathbf{T}}$ is a $\mathbb{P}$-martingale if $R_1, \ldots, R_T$ are independent and $\mathbb{E}[R_t] = 0$.

ii) Give necessary and sufficient conditions on $R_1, \ldots, R_T$ such that $(S_t)_{t \in \mathbf{T}}$ is a $\mathbb{P}$-martingale.

iii) Do you find an example such that $(S_t)_{t \in \mathbf{T}}$ is a martingale, but the returns are not independent?