Portfolio Theory - Exercise Set 1

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1 Financial Markets in Finite Discrete-Time

We assume that $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, X)$ is a frictionless financial market in finite discrete-time $\mathbf{T} = \{0, 1, \ldots, T\}$ and $(d+1)$-tradable assets with $d \in \mathbb{N}$. If φ_t and X_t are two $(d+1)$-dimensional vectors we denote the scalar product as $\varphi_t^\intercal X_t$, but you might also write $\varphi_t \cdot X_t$ or $\langle \varphi_t, X_t \rangle_{\mathbb{R}^d}$ if you prefer. For any stochastic process $(Y_t)_{t\in\mathbf{T}}$ we write $\Delta Y_t = Y_t - Y_{t-1}$ for all $t \geq 1$ and set $\Delta Y_0 = Y_0$. For $A, B \in \mathcal{F}$ we use the notation $\mathbb{P}(B|A) = \mathbb{P}(B \cap A)/\mathbb{P}(A)$ to denote the conditional probability.

Now, solve the following exercises:

1.1 (Predictable or Adapted? [2p])
Consider a financial market with only two assets which are modeled as usual by the stochastic process $X = (S^{(0)}, S^{(1)})$ and that is adapted to the filtration $(\mathcal{F}_t)_{t\in\mathbf{T}}$.

Decide which of the following processes φ are predictable and which in general are not:

(a) $\varphi_t = \mathbf{1}_{\{S_t^{(1)} > S_{t-1}^{(1)}\}}$;

(b) $\varphi_1 = 1$ and $\varphi_t = \mathbf{1}_{\{S_{t-1}^{(1)} > S_{t-2}^{(1)}\}}$ for $t \geq 2$;

(c) $\varphi_t = \mathbf{1}_A \cdot \mathbf{1}_{\{t > t_0\}}$, where $t_0 \in \{0, \ldots, T\}$ and $A \in \mathcal{F}_{t_0}$;

(d) $\varphi_t = \mathbf{1}_{\{S_t^{(1)} > S_0^{(1)}\}}$;

(e) $\varphi_1 = 1$ and $\varphi_t = 2\varphi_{t-1}\mathbf{1}_{\{S_{t-1}^{(1)} < S_0^{(1)}\}}$ for $t \geq 2$.

1.2 (Discounting and self-financing strategies [2p])
Prove Lemma 1.10 of the lecture notes.

1.3 (Equivalent definition of self-financing strategies [2p])
Let $\varphi = (\varphi_t)_{t\in\mathbf{T}}$ be a trading strategy such that $\varphi_t^\intercal X_t = \varphi_{t+1}^\intercal X_t$ for all $t \in \{0, 1, \ldots, T-1\}$. Show that this is equivalent to

$$\Delta W_t(\varphi) = \varphi_t^\intercal \Delta X_t, \quad \forall t \in \{1, 2, \ldots, T\},$$

where $(W_t(\varphi))_{t\in\mathbf{T}}$ denotes the wealth process of the strategy φ. Whenever a trading stratefy satisfies one of the equivalent conditions, we call the strategy φ *self-financing*.

1.4 (Martingale measures and densities [3p])
Let $\mathbb{Q}$ be a probability measure on $(\Omega, \mathcal{F})$ which is absolutely continuous with respect to $\mathbb{P}$. Show that the density process

$$Z_t \coloneqq \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}|_{\mathcal{F}_t}, \quad t = 0, 1, \ldots, T,$$

is a martingale with respect to $\mathbb{P}$, where $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}|_{\mathcal{F}_t}$ denotes the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ restricted to the sub-σ-algebra $\mathcal{F}_t$.

1.4 (A geometric model [3p])
Let $(S_t)_{t\in\mathbf{T}}$ be a strictly positive price process of some risky asset, say a stock. The returns are defined by $R_t = \Delta S_t / S_{t-1}$ for $t = 1, 2, \ldots, T$ so that $S_t = S_0 \prod_{k=1}^t (1 + R_k)$ and let the filtration $\mathbb{F}$ be given by $\mathcal{F}_t = \sigma(S_0, \ldots, S_t)$. Then:

i) Show that $(S_t)_{t\in\mathbf{T}}$ is a $\mathbb{P}$-martingale if $R_1, \ldots, R_T$ are independent and $\mathbb{E}[R_t] = 0$.

ii) Give necessary and sufficient conditions on $R_1, \ldots, R_T$ such that $(S_t)_{t\in\mathbf{T}}$ is a $\mathbb{P}$-martingale.

iii) Do you find an example such that $(S_t)_{t\in\mathbf{T}}$ is a martingale, but the returns are not independent?

1.5 (Equivalent Martingale Measure [3p])

Let $Z_1, \ldots, Z_T$ be independent standard normal random variables on $(\Omega, \mathcal{F}, P)$, and let $\mathcal{F}_t$ be the σ-field generated by $Z_1, \ldots, Z_t$, where $t = 1, \ldots, T$. We also let $\mathcal{F}_0 := \{\emptyset, \Omega\}$. For constants $X_0^1 > 0$, $\sigma_i > 0$, and $m_i \in \mathbb{R}$, we now define the discounted price process of a risky asset as the following sequence of log-normally distributed random variables,

$$X_t^{(1)} := X_0^{(1)} \prod_{i=1}^{t} e^{\sigma_i Z_i + m_i}, \quad t = 0, \ldots, T. \tag{1}$$

Construct an equivalent martingale measure for $X^{(1)}$ under which the random variables $X_t^{(1)}$ have still a log-normal distribution.

1.7 (Insider Trading [3p])

Let $Y_1, \ldots, Y_T$ be *iid* random variables on some $(\Omega, \mathcal{F}, P)$ with $\mathbb{E}[Y_t] = 0$ for all t and let $\mathcal{F}_t = \sigma(Y_1, \ldots, Y_t)$. Let $X_t = \sum_{k=1}^{t} Y_k$ for $t \leq T$. Obviously, the X_t form a martingale. Consider an insider trader, that is a trader whose information pattern is given by the σ-algebras $\widetilde{\mathcal{F}}_t := \sigma(\mathcal{F}_t \cup \sigma(X_T))$, i.e. at any time $t \leq T$ she 'knows' the final value X_T.

(a) Show that the X_t don't result in a martingale w.r.t. the enlarged filtration of the $\widetilde{\mathcal{F}}_t$.

(b) Let $\widehat{X}_t = X_t - \frac{\sum_{k=0}^{t-1} X_T - X_k}{T-k}$, $t \leq T$. Show that the $(\widehat{X}_t)_{t \in \mathbb{T}}$ yield a martingale w.r.t. enlarged filtration. (Hint: use that $\mathbb{E}[X_t | X_T] = \frac{t}{T} X_T$ and independence of the Y_k.)

(c) Construct a self-financing strategy of investments ξ_t w.r.t. the enlarged filtration (so the $\tilde{\xi}_t$ are $\widetilde{\mathcal{F}}_{t-1}$-measurable) such that $\mathbb{E}[\sum_{t=1}^{T} \xi_t (X_t - X_{t-1})]$ is positive. This should follow from maximization of the expected gain $\mathbb{E}[\sum_{t=1}^{T} \tilde{\xi}_t (X_t - X_{t-1})]$ over all self-financing strategies such that $|\tilde{\xi}_t| \leq 1$.