



Risk Measures

10.1 (Axiomatic description of risk [3p])

Let ρ be a normalized risk measure on \mathcal{X} . Show that any two of the following properties imply the remaining third:

- i) Convexity, i.e. $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$ for $\lambda \in [0, 1]$ and $X, Y \in \mathcal{X}$;
- ii) Positive homogeneity, i.e. $\rho(\lambda X) = \lambda\rho(X)$ for all $\lambda \geq 0$ and $X \in \mathcal{X}$;
- iii) Subadditivity, i.e. $\rho(X + Y) \leq \rho(X) + \rho(Y)$ for $X, Y \in \mathcal{X}$.

10.2 (Value-at-Risk [3p])

Consider the Value-at-Risk risk measure VaR_λ from Example 4.7. This is not a convex risk measure, but it can be transformed to one as follows:

We define the **Average Value at Risk** at level $\lambda \in (0, 1]$ of a financial position $X \in \mathcal{X}$ as

$$\text{AVaR}_\lambda(X) := \frac{1}{\lambda} \int_0^\lambda \text{VaR}_\alpha(X) d\alpha.$$

The average value at risk $\text{AVaR}_\lambda(X)$ has the following representation, that you don't have to prove:

$$\text{AVaR}_\lambda(X) = \frac{1}{\lambda} \mathbb{E}_{\mathbb{P}} [(q - X)^+] - q = \frac{1}{\lambda} \inf_{r \in \mathbb{R}} (\mathbb{E}_{\mathbb{P}} [(r - X)^+] - \lambda r),$$

for any λ -quantile q of X . Use this and Exercise 10.1 to show that the average value at risk is a coherent risk measure on \mathcal{X} .

10.3 (Gini's mean difference [3p])

For any $X \in L_1$ define $\Delta(X) := \mathbb{E}[|X - \tilde{X}|]$, where \tilde{X} is an independent copy of X . The function $\Delta(\cdot)$ is often called **Gini's mean difference** and satisfies $\Delta(X) \in [0, 2\mathbb{E}[|X|]]$, with $\Delta(X) = 0$ if and only if X is P -a.s. constant. In analogy to the mean-standard deviation risk measure in Example 4.9, consider the functional on L_1 defined by

$$\rho_\lambda(X) := \mathbb{E}[-X] + \lambda\Delta(X), \quad \text{for some } \lambda \geq 0.$$

- (a) Show that $\rho_\lambda(X)$ is a coherent risk measure for any $\lambda \in [0, \frac{1}{2}]$. Note that a non-constant position X is acceptable if and only if $\mathbb{E}[X] > 0$ and the Gini coefficient of X , defined as

$$G(X) := \frac{\Delta(X)}{2\mathbb{E}[X]},$$

satisfies $G(X) \geq (2\lambda)^{-1}$.

- (b) Show that for $\lambda > \frac{1}{2}$ there are nonnegative random variables X such that $\rho_\lambda(X) > 0$. In particular, ρ_λ cannot be monotone for $\lambda > \frac{1}{2}$.
- (c) Show that

$$\Delta(X) = 2 \int_{-\infty}^{\infty} F(x)(1 - F(x)) dx,$$

where F denotes the distribution function of X , and that

$$\Delta(X) = 4 \text{cov}(X, F(X))$$

whenever X has a continuous distribution.

- (d) Use the above equation to show that $\Delta(X) \leq \frac{2\sigma(X)}{\sqrt{3}}$ for all $X \in L_2$ and conclude that ρ_λ is dominated by the mean-standard deviation risk measure $\rho_{RC}(X) = -\mathbb{E}[X] - c\sigma^2(X)$ if $\lambda \leq \frac{c\sqrt{3}}{2}$. As in Example 4.9, we denote here by $\sigma(X)$ the square root of the variance of X .
- (e) Compute $\Delta(X)$ and $G(X)$ if X has a Pareto distribution with shape parameter $\alpha > 1$ and minimum 1, that is, log X is exponentially distributed with parameter α .
- (f) Consider a log-normally distributed random variable $X = \exp(m + \sigma Z)$, where Z has a standard normal law $N(0, 1)$. Show that $G(X) = \text{erf}(\frac{\sigma}{2})$, where $\text{erf}(\cdot)$ is the Gaussian error function, that is,

$$\text{erf}(z) := \mathbb{P}[|Z| \leq z\sqrt{2}] = 2\Phi(z\sqrt{2}) - 1.$$