Portfolio Theory - Exercise Set 11

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Optimal Control and Dynamic Programming

11.1 (Maximization of expected utility from terminal wealth in the CRR model [3p])
Consider a CRR model, in which the returns are *iid* with

$$P(R_t = U) = p \quad \text{(where } p \text{ is not necessarily equal to the risk neutral value } p^*\text{)}.$$

Consider the maximization of the expected utility of terminal wealth $\mathbb{E}[u(W_T)]$, with $u(x) = \log x$. Compute the optimal trading strategy φ^* via dynamic programming.

(Hint: show that $\hat{v}_t(x) = \log x + k_t$ for some constants k_t and $\hat{v}_t$ as in equation (93) of the lecture notes.)

11.2 (Maximal wealth in the CRR model [3p])

Consider a CRR model as before with a parameter p that determines the probability measure $\mathbb{P}$. Let $p^* := (r - D)(U - D)^{-1}$ (see Theorem 2.16), and again assume that $u(x) = \log x$. Now, we fix the initial capital as $W_0 := w$.

(a) Show that the optimal attainable terminal wealth is given by

$$W_T = w(1+r)^T \left(\frac{p}{p^*}\right)^{B_T} \left(\frac{1-p}{1-p^*}\right)^{T-B_T},$$

where B_T is the number of 'up-movements' of the stock.

(b) Assume that at time $T-1$ and that B_{T-1} 'up-movements' have been observed. Show, using the risk-neutral approach, that for the optimal replicating strategy one has

$$\varphi_T^{(1)} = w(1+r)^T \left(\frac{p}{p^*}\right)^{T-1-B_{T-1}} \frac{p - p^*}{S_{T-1}(U-D)p^*(1-p^*)}$$

and

$$\varphi_T^{(0)} = w \left(\frac{p}{p^*}\right)^{B_{T-1}} \left(\frac{1-p}{1-p^*}\right)^{T-1-B_{T-1}} \left(\frac{p^* - p + Up^*(1-p) - Dp(1-p^*)}{(U-D)p^*(1-p^*)}\right).$$

(c) Show that for the optimal wealth we have

$$W_{T-1} = w(1+r)^{T-1} \left(\frac{p^*}{p^*}\right)^{B_{T-1}} \left(\frac{1-p}{1-p^*}\right)^{T-1-B_{T-1}},$$

and that the fraction of the wealth W_{T-1} that is invested in the risky asset is equal to

$$\frac{(1+r)(p-p^*)}{(U-D)p^*(1-p^*)}.$$

(d) Conjecture what the fraction of the capital W_t is, that is invested in the risky asset at $t < T-1$.

11.3 (Optimality Principle [3p])
Prove the "Optimality Principle" Proposition 7.6.