Portfolio Theory - Exercise Set 2

Contact: s.karbach@uva.nl // Submit until 20.09.2023

Exercises: Arbitrage

We assume that $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, X)$ is a frictionless discounted financial market in finite discrete-time $\mathbf{T} = \{0, 1, \dots, T\}$ and $(d+1)$-tradable assets with $d \in \mathbb{N}$. For $A, B \in \mathcal{F}$ we use the notation $\mathbb{P}(B|A) = \mathbb{P}(B \cap A)/\mathbb{P}(A)$ to denote the conditional probability.

Now, solve the following exercises:

2.1 (Arbitrage [3p])
Let φ be a self-financing strategy and assume that the market is arbitrage-free. Show that the following implications hold for all $t = 0, 1, \dots, T-1$ and $A \in \mathcal{F}_t$ with $\mathbb{P}(A) > 0$:

$$\begin{aligned} &1) \quad \mathbb{P}(W_{t+1}(\varphi) - W_t(\varphi) \geq 0|A) = 1 \quad \Longrightarrow \quad \mathbb{P}(W_{t+1}(\varphi) - W_t(\varphi) = 0|A) = 1, \\ &2) \quad \mathbb{P}(W_{t+1}(\varphi) - W_t(\varphi) \leq 0|A) = 1 \quad \Longrightarrow \quad \mathbb{P}(W_{t+1}(\varphi) - W_t(\varphi) = 0|A) = 1. \end{aligned}$$

2.2 (Equivalent Martingale Measure [3p])
Let $Z_1, \dots, Z_T$ be independent standard normal random variables on $(\Omega, \mathcal{F}, P)$, and let $\mathcal{F}_t$ be the σ-field generated by $Z_1, \dots, Z_t$, where $t = 1, \dots, T$. We also let $\mathcal{F}_0 := \{\emptyset, \Omega\}$. For constants $X_0^1 > 0$, $\sigma_i > 0$, and $m_i \in \mathbb{R}$, we now define the discounted price process of a risky asset as the following sequence of log-normally distributed random variables,

$$X_t^{(1)} := X_0^{(1)} \prod_{i=1}^{t} e^{\sigma_i Z_i + m_i}, \quad t = 0, \dots, T. \tag{1}$$

Construct an equivalent martingale measure for $X^{(1)}$ under which the random variables $X_t^{(1)}$ have still a log-normal distribution.

2.3 (Insider Trading [3p])
Let $Y_1, \dots, Y_T$ be *iid* random variables on some $(\Omega, \mathcal{F}, P)$ with $\mathbb{E}[Y_t] = 0$ for all t and let $\mathcal{F}_t = \sigma(Y_1, \dots, Y_t)$. Let $X_t = \sum_{k=1}^{t} Y_k$ for $t \leq T$. Obviously, the X_t form a martingale. Consider an insider trader, that is a trader whose information pattern is given by the σ-algebras $\widetilde{\mathcal{F}}_t := \sigma(\mathcal{F}_t \cup \sigma(X_T))$, i.e. at any time $t \leq T$ she 'knows' the final value X_T.

(a) Show that the X_t don't result in a martingale w.r.t. the enlarged filtration of the $\widetilde{\mathcal{F}}_t$.

(b) Let $\widetilde{X}_t = X_t - \frac{\sum_{k=0}^{t-1} X_T - X_k}{T-k}$, $t \leq T$. Show that the $(\widetilde{X}_t)_{t \in \mathbb{T}}$ yield a martingale w.r.t. enlarged filtration. (Hint: use that $\mathbb{E}[X_t|X_T] = \frac{t}{T} X_T$ and independence of the Y_k.)

(c) Construct a self-financing strategy of investments ξ_t w.r.t. the enlarged filtration (so the $\tilde{\xi}_t$ are $\widetilde{\mathcal{F}}_{t-1}$-measurable) such that $\mathbb{E}[\sum_{t=1}^{T} \xi_t (X_t - X_{t-1})]$ is positive. This should follow from maximization of the expected gain $\mathbb{E}[\sum_{t=1}^{T} \tilde{\xi}_t (X_t - X_{t-1})]$ over all self-financing strategies such that $|\tilde{\xi}_t| \leq 1$.