Portfolio Theory - Exercise Set 3

Contact: s.karbach@uva.nl // Submit until 27.09.2023

Exercises: Pricing and Hedging European Contingent Claims

Read Definition 2.7 in the course lecture notes to get yourself familiar with the notion of an arbitrage-free price (this is required for Exercise 3.2 and 3.3 below). Then solve the following exercises:

3.1 (Parametrization of Martingale Measures [3p])

Fix the time horizon at T and assume the initial σ-algebra $\mathcal{F}_0$ to be trivial. Let $(S_t^{(0)})_{t\in\mathbf{T}}$ be identically equal to 1 and let $Z_t := \log \frac{S_t}{S_{t-1}}$. Suppose that the market that is described by the pair of processes $S^{(0)}, S^{(1)}$ is arbitrage-free. Suppose that $\mathbb{P}$ is such that the $Z_1, Z_2, \ldots, Z_T$ are i.i.d. with a common normal $\mathcal{N}(\mu, \sigma^2)$ distribution.

Give a relation between the parameters μ and σ^2 if $\mathbb{P}$ is a martingale measure. Is it possible that Z_t has a Gamma distribution instead of a normal one if $\mathbb{P}$ is a martingale measure?

3.2 (Sensitivity of option prices [3p])

Consider an arbitrage-free market with one risky asset. Let $S^{(1)}$ be its price process and $S^{(0)}$ the deterministic price process of the riskless asset. Consider a European call option with discounted payoff

$$\widetilde{H} = \frac{(S_T^{(1)} - K)^+}{S_T^{(0)}},$$

for some $K > 0$. Assume that $S_T^{(1)}$ has a density w.r.t. Lebesgue measure under any risk-neutral measure. Let π^* be an arbitrage-free price of the call option under some risk-neutral measure $\mathbb{P}^*$. Obviously π^* depends on K and $S_0^{(1)}$, so we write $\pi^* = \pi^*(K, S_0^{(1)})$. Show that

$$\frac{\partial \pi^*}{\partial S_0^{(1)}} < 1, \quad \frac{\partial \pi^*}{\partial K} = -\left(1 - F^*(K)\right)\frac{1}{S_T^{(0)}},$$

where F^* is the distribution function of $S_T^{(1)}$ under P^*. To show the first assertion you may make additional assumptions, e.g. that $S_T^{(1)}$ is increasing in $S_0^{(1)}$, or even more explicit, $S_T^{(1)} = S_0^{(1)} R_T$, with R_T a positive random variable.

3.3 (Put-call parity in a multi-period model [3p])

Consider an arbitrage-free market model with a single risky asset, $S^{(1)}$, and a riskless bank account, $S_t^{(0)} = (1+r)^t$, for some $r > -1$. Suppose that an arbitrage-free price π_{call} has been fixed for the discounted claim

$$H_{\text{call}} = \frac{(S_T^{(1)} - K)^+}{S_T^{(0)}}$$

of a European call option with strike $K \geq 0$. Then there exists a nonnegative adapted process $X^{(2)}$ with $X_0^{(2)} = \pi_{\text{call}}$ and $X_T^{(2)} = H_{\text{call}}$ such that the extended market model with discounted price process $(\mathbb{1}, \widetilde{S}^{(1)}, \widetilde{X}^{(2)})$ is arbitrage-free.

Show that the discounted European contingent claim

$$H_{\text{put}} = \frac{(K - S_T^{(1)})^+}{S_T^{(0)}}$$

is attainable in the extended model, and that its unique arbitrage-free price is given by

$$\pi_{\text{put}} = \frac{K}{(1+r)^T} - S_0^{(1)} + \pi_{\text{call}}.$$