Portfolio Theory - Exercise Set 4

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Exercises: Complete Markets and the Standard Model

Complete Markets

4.1 (Additional Information in the Market [3p])

Let's examine a complete market model with finite time horizon $T = 2$ and a discounted price process $(X_t)_{t=0,1,2}$, on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,1,2}, \mathbb{P})$. We denote the unique equivalent martingale measure as $\mathbb{Q}$.

Now, we extend this model by introducing two additional states, ω^+ and ω^-, describing further information that will be uncovered at time $t = 2$. We then define:

$$\begin{aligned}\widehat{\Omega} &:= \Omega \times \{\omega^+, \omega^-\},\\ \widehat{\mathcal{F}} &:= \sigma(A \times B | A \in \mathcal{F},\, B \in \{\omega^+, \omega^-\}),\\ \widehat{\mathbb{P}}[\{\omega, \omega^\pm\}] &:= \frac{1}{2}\mathbb{P}[\{\omega\}], \quad \omega \in \Omega.\end{aligned}$$

The unveiling of additional information at $t = 2$ is reflected by the chosen price process $\hat{X}_t(\omega, \omega^\pm) := X_t(\omega)$ and the following filtration:

$$\begin{aligned}\widehat{\mathcal{F}}_0 &:= \{\emptyset, \widehat{\Omega}\},\\ \widehat{\mathcal{F}}_1 &:= \sigma(A \times \{\omega^+, \omega^-\} | A \in \mathcal{F}_1),\\ \widehat{\mathcal{F}}_2 &:= \widehat{\mathcal{F}}.\end{aligned}$$

For $0 < p < 1$, we further specify the probability measure $\hat{\mathbb{P}}^*_p$ on $\widehat{\mathcal{F}}$ via

$$\widehat{\mathbb{P}}^*_p[A \times \{\omega^+\}] \coloneqq p \cdot \mathbb{Q}[A], \quad \widehat{\mathbb{P}}^*_p[A \times \{\omega^-\}] \coloneqq (1-p) \cdot \mathbb{Q}[A] \quad A \in \mathcal{F}.$$

Then solve the following exercises:

(a) Verify that each measure $\hat{\mathbb{P}}^*_p$ is an equivalent martingale measure for the extended model.

(b) Argue that the extended model is incomplete and find a nonattainable contingent claim.

(c) Does every equivalent martingale measure belong to the set $\{\widehat{\mathbb{P}}^*_p \colon 0 < p < 1\}$? If yes, give a proof, if not, find a counterexample.

4.2 (Arbitrage-free prices of non-attainable claims [3p])

Prove that the measure $\check{\mathbb{Q}}$ in the proof of Theorem 2.9 is a martingale measure for the market $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, X)$.

(there is one more exercise on the next page)

The standard model

Consider a frictionless financial market $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, X)$ with two assets represented by $X = (S^{(0)}, S^{(1)})$. The process of the bank account is given as $S_t^{(0)} = S_0^{(0)} \mathrm{e}^{rt} = S_0^{(0)}(1+\tilde{r})^t$, $t = 0, 1, \ldots, T$, with fixed interest rate $r \in \mathbb{R}$ and $\tilde{r} = \mathrm{e}^r - 1$; and the risky asset's price follows the model:

$$S_t^{(1)} = S_0^{(1)} \exp(R_t) = S_0^{(1)} \prod_{j=1}^{t} (1 + \Delta \widetilde{R}_j), \tag{1}$$

where $\widetilde{R}_j = \sum_{n=1}^{j} (\exp(\Delta R_n) - 1)$, $R_0 = 0$, and the *daily logarithmic returns* $\Delta R_1, \Delta R_2, \ldots$ of $(R_t)_{t \in \mathbf{T}}$ (note that $\Delta R_t = \log(S_t^{(1)}/S_{t-1}^{(1)})$ for $t = 1, \ldots, T$) are assumed to be independent and identically distributed.

We call the market $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, X)$ the *standard model*, whenever the daily logarithmic returns ΔR_t for $t = 1, \ldots, T$ are normally distributed with mean μ and variance σ^2. Note that according to Proposition 2.13 in the lecture notes, the standard model can not be complete as the normal distribution is a continuous distribution and hecne, the underyling probability space can not be decomposed into a finite number of atoms!

4.2 (European Call Option in the Standard Model [3p])
Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, X)$ be the *standard model* introduced above. Give a solution to the following problems:

(a) Give an estimator for the parameters μ and σ^2 at time T given the historic data $S_0^{(1)}, S_1^{(1)}, \ldots, S_{T-1}^{(1)}$.

(b) Consider an European Call Option with maturity T and strike K, i.e., an European contingent claim with pay-off at time T given by $H := (S_T^{(1)} - K)^+$.

i) Construct a trading strategy φ such that $W_T(\varphi) \geq H$ $\mathbb{P}$-almost surely.

ii) Let $\pi_L(H) := \sup\{W_0 \in \mathbb{R} \colon \exists \text{ self-financing strategy } \varphi \text{ s.t. } W_0(\varphi) = W_0 \text{ and } W_T(\varphi) \geq H\}$. Show that:

$$\pi_L(H) \geq \max(0, S_0^{(1)} - K\mathrm{e}^{-rT}) = (S_0^{(1)} - K\mathrm{e}^{-rT})^+.$$

(c) Note that the two distributions $\mathcal{N}(\mu, \sigma^2)$ and $\mathcal{N}(r - \frac{1}{2}\tilde{\sigma}^2, \tilde{\sigma}^2)$ are equivalent to the Lebesgue measure and thus also equivalent to each other with density denoted by

$$f_{\tilde{\sigma}} := \frac{\mathrm{d}\mathcal{N}(r - \frac{1}{2}\tilde{\sigma}^2, \tilde{\sigma}^2)}{\mathrm{d}\mathcal{N}(\mu, \sigma^2)}.$$

Next, define a probability measure $\mathbb{Q}_{\tilde{\sigma}} \sim \mathbb{P}$ by

$$\frac{\mathrm{d}\mathbb{Q}_{\tilde{\sigma}}}{\mathrm{d}\mathbb{P}} := \prod_{n=1}^{T} f_{\tilde{\sigma}}(\Delta R_n).$$

Relative to the measure $\mathbb{Q}_{\tilde{\sigma}}$, the price process has the same structure as under $\mathbb{P}$, but with different parameters μ and σ^2. Show that for all $\tilde{\sigma} > 0$ the following holds true:

i) $\mathbb{Q}_{\tilde{\sigma}}$ is a probability measure equivalent to $\mathbb{P}$ and the random variables $\Delta R_1, \ldots, \Delta R_N$ are independent under $\mathbb{Q}_{\tilde{\sigma}}$ with law $\mathcal{N}(r - \tilde{\sigma}^2/2, \tilde{\sigma}^2)$;

ii) $\mathbb{Q}_{\tilde{\sigma}}$ is an equivalent martingale measure.

Note that this then constructs a one-parametric family of equivalent martingale measures $\{\mathbb{Q}_{\tilde{\sigma}} \colon \tilde{\sigma} > 0\}$.

(d) Next, compute the option price that is obtained as an expectation under measure $\mathbb{Q}_{\tilde{\sigma}}$, i.e.,

$$\pi_{\mathbb{Q}_{\tilde{\sigma}}} := S_0^{(0)} \mathbb{E}_{\mathbb{Q}_{\tilde{\sigma}}} \left(\frac{(S_T^{(1)} - K)^+}{S_T^{(0)}} \right) = \mathbb{E}_{\mathbb{Q}_{\tilde{\sigma}}} \left((S_0^{(1)} \mathrm{e}^{R_T - rT} - K\mathrm{e}^{-rT})^+ \right).$$

(e) The options price $\pi_{\mathbb{Q}_{\tilde{\sigma}}}$ depends via $\mathbb{Q}_{\tilde{\sigma}}$ on the parameter $\tilde{\sigma}$. Show that if $\tilde{\sigma} \to 0$ then $\pi_{\mathbb{Q}_{\tilde{\sigma}}} \to (S_0^{(1)} - K\mathrm{e}^{-rT})^+$ and if $\sigma \to \infty$ then $\pi_{\mathbb{Q}_{\tilde{\sigma}}} \to S_0^{(1)}$. (Therefore, the price limits in the standard model coincide with the trivial ones. Put differently, absence of arbitrage does not provide much information on European call prices in the standard model.)