Exercises: The Cox-Ross-Rubinstein Model

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}, (S^{(0)}, S^{(1)}))$ denote a canonicl CRR model which is complete. Solve the following exercises:

4.1 (Forward-start Options [3p])
Let $T_0 \in \{1, \ldots, T-1\}$ and $K > 0$. The payoff of *forward starting call option* has the form

$$\left(\frac{S_T^{(1)}}{S_{T_0}^{(1)}} - K \right)^+ .$$

Determine its arbitrage-free price and replicating strategy in the CRR model.

4.2 (One-period CRR [3p])
Lets assume that $T = 1$, i.e., we assume a one-period CRR model. Suppose we want to determine the price at time zero of the derivative $H = S_1^{(1)}$, i.e., the derivative pays off the stock price at time $T = 1$. What is the time-zero price W_0^H given by the risk-neutral pricing formula?

4.3 (Asian Option [3p])
Consider the three-period CRR model in Figure 1 below and take the interest rate $r = 0.25$. What is D, U, $\mathbb{Q}(R_t = U)$ in this case? For $n = 0, 1, 2, 3$ define

$$Y_n = \sum_{k=0}^{n} S_k^{(1)},$$

the sum of the stock prices between times zero and n. Consider an Asian call option, see Example 2.4 in the lecture notes, that expires at time three and has strike $K = 4$, i.e., whose payoff at time $T = 3$ is

$$H^{\text{asian}} = (\frac{1}{4} Y_3 - 4)^+.$$

Let $W_n^{\text{asian}}(s, y)$ denote the price of this option at time n, if $S_n^{(1)} = s$ and $Y_n = y$. In particular, we have $W_3^{\text{asian}}(s, y) = (\frac{1}{4} y - 4)^+$.

(a) Develop an algorithm for computing W_n^{asian} recursively. In particular, write a formula for W_n^{asian} in terms of W_{n+1}^{asian}.

(b) Apply the algorithm developed in (i) to compute $W_0^{\text{asian}}(4, 4)$, the price of the Asian option at time zero.

(c) Provide a formula for $\delta_n(s, y)$, the number of shares of stock that should be held by the replicating portfolio at time n if $S_n^{(1)} = s$ and $Y_n = y$.

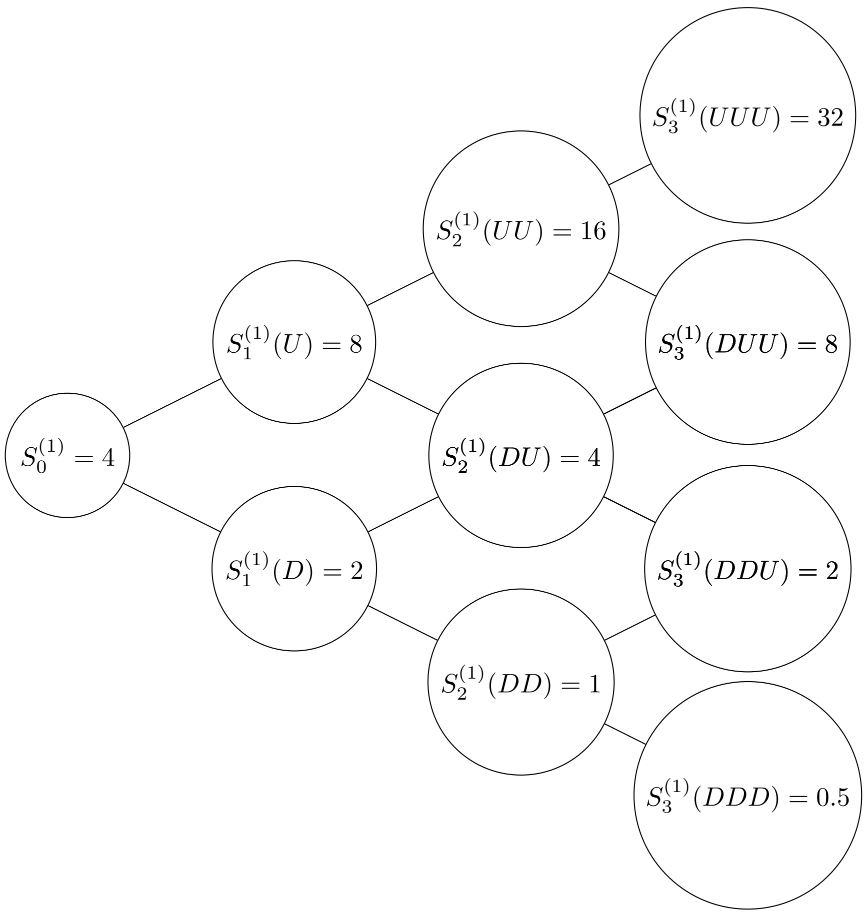

Figure 1: Three-period binomial asset pricing model.