Portfolio Theory - Exercise Set 7

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Exercises: Expected Utility

7.1 (An utility inequality [3p])
Show that for a utility function $u \in C^1(\mathbb{R})$ it holds that

$$m(\mu) > c(\mu) > \frac{\mathbb{E}\left[Xu'(X)\right]}{\mathbb{E}\left[u'(X)\right]},$$

where X has nondegenerate distribution μ and all expectations are assumed to be finite.

7.2 (CARA utility function [3p])
Let $u(x) = 1 - \exp(-x)$, a CARA function. Consider an investor with utility function u who wants to invest an initial capital. There is one riskless asset, having value 1 and interest rate $r = 0$, and one risky assets with random pay-off S_1 having a normal $\mathcal{N}(m, \sigma^2)$ distribution with $\sigma^2 > 0$. Suppose he invests a fraction λ in the riskless asset and the remainder in the risky asset. The pay-off of this portfolio is thus $\lambda + (1 - \lambda)S_1$. The aim is to maximize his expected utility.

(a) Show that $\mathbb{E}\left[\exp(uS_1)\right] = \exp(um + \frac{1}{2}u^2\sigma^2)$ $(u \in \mathbb{R})$.

(b) Compute for each λ the certainty equivalent of the portfolio.

(c) Let λ^* be the optimal value of λ. Give, by direct computations, conditions on the parameters such that each of the cases $\lambda^* = 0$, $\lambda^* = 1$ or $\lambda^* \in (0, 1)$ occurs.

(d) Compare the results of (c) with the assertions of Proposition 3.23.

7.3 (Decreasing risk aversion [3p])
A utility function $u : \mathbb{R} \to \mathbb{R}$ is said to exhibit *decreasing risk aversion* if the function $x \mapsto \alpha(x)$ (the Arrow-Pratt coefficient) is decreasing. Show that this property is equivalent to saying that for every $x_1 < x_2$ there exists a concave function g such that $g(u(x_1)) < g(u(x_2))$.