Portfolio Theory - Exercise Set 9

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9.1 (Variance-optimal hedge under martingale measures [3p])
Note that in Section 6.3 we let X denote the already discounted asset price process. In the first exercise we fill in the open gaps in the lecture notes. Indeed, solve the following:

i) Prove the remaining part of Lemma 6.15, i.e., show that

a) the process $(M_t X_t)_{t\in\mathbf{T}}$ is a martingale;

b) the Kunita-Watanabe decomposition in equation (83) is unique.

ii) Prove Theorem 6.16.

(Hint: It might help to use the predictable quadratic (co)variation process $\langle M\rangle$ for square-integrable martingales M given by $\Delta\langle M\rangle = \mathbb{E}_{\mathbb{P}}\left[(M_n - M_{n-1})^2|\mathcal{F}_{n-1}\right]$ and its properties for the derivations (see Section 9 in the MTP lecture notes).

9.2 (Variance-optimal hedge in an affine GARCH model [3p])
In this exercise we consider a univariate discrete-time stochastic volatility model of GARCH type given as follows: we model the discounted underlying asset price process $(\widetilde{S}_t)_{t\in\mathbf{T}}$ as

$$\widetilde{S}_t = \widetilde{S}_{t-1}\exp\big(-\frac{1}{2}V_t + \sqrt{V_t}z_t^*\big), \tag{1}$$

$$V_t = \omega + \beta V_{t-1} + \alpha\big(z_{t-1}^* - \gamma^*\sqrt{V_{t-1}}\big)^2, \tag{2}$$

for some suitable parameters ω, α, β and γ such that $V_t \geq 0$ for all $t \in \mathbf{T}$ and where z_t^* is standard normal distributed. The process $(V_t)_{t\in\mathbf{T}}$ is called the *instantaneous volatiliy process* of $\widetilde{S}$. We also assume that the discounted asset price process $(\widetilde{S}_t)_{t\in\mathbf{T}}$ is square-integrable with positive conditional variance process $(\sigma_t^2)_{t=1,2,\dots,T}$ and we denote by $\widetilde{H}$ some discounted square-integrable contingent claim.

i) Argue why a variance-optimal strategy (W_0^*, ϕ^*) for $\widetilde{H}$ exists and provide an expression of the strategy using Theorem 6.16.

ii) Under the additional assumption that $H = f(\widetilde{S}_T)$ for some function f, we have a integral representation for $f\colon \mathbb{C}\to\mathbb{C}$ of the form

$$f(x) = \int_{R-\mathrm{i}\,\infty}^{R+\mathrm{i}\,\infty} x^u l(u)\,\mathrm{d}u,$$

for some function l and $R \in \mathbb{R}$. For instance, the payoff of an European Call Option can be written as

$$f(x) = (x-K)^+ = \frac{1}{2\pi\mathrm{i}}\int_{R-\mathrm{i}\,\infty}^{R+\mathrm{i}\,\infty} x^u \frac{K^{1-u}}{u(u-1)}\,\mathrm{d}u.$$

a) Assume that $\widetilde{H}$ has an integral representation as above. Then show that the derivative prices $\widetilde{W}_t^H$ for $t = 0, 1, \dots, T-1$ under some pricing measure $\mathbb{Q}$ and the variance-optimal hedge under the same measure can be expressed using such complex integrals as well.

b) Take as a fact that the model (1)-(2) is *affine*, which means that the joint moment-generating function $g(t, T, u, v)$ of $(\widetilde{S}_t, V_t)$ for any $t \in \mathbf{T}$ has an exponential affine form:

$$g(t,T,u,v) = \mathbb{E}_{\mathbb{Q}}\left[\widetilde{S}_T^u \exp\big(v\sum_{k=1}^{n} h_{t+k}\big)|\mathcal{F}_t\right] = S_t^u\exp(A(t,T,u,v) + B(t,T,u,v)h_{t+1}),$$

for two deterministic functions A and B solving some associated difference equations. Use the representation in a) and this fact to show that

$$\widetilde{W}_t^H = \int_{R-\mathrm{i}\,\infty}^{R+\mathrm{i}\,\infty} g(t;T,u,0)l(u)\,\mathrm{d}u,$$

and that the variance-optimal hedge is given by

$$\phi_t^* = \int_{R-\mathrm{i}\,\infty}^{R+\mathrm{i}\,\infty} \frac{g(t,T,u+1,0) - \widetilde{S}_t g(t,T,u,0)}{g(t,T,2,0) - \widetilde{S}_t^2} l(u)\,\mathrm{d}u.$$

In the next exercise we construct an example of a financial market, where the bounded mean-variance trade-off condition (81) in the lecture notes is not satisfied and where the subspace $\mathcal{G}_T$ is indeed *not* closed.

9.3 (Counterexample for closedness of the space $\mathcal{G}_T$ [3p])
Let $\Omega = [0,1] \times \{-1,+1\}$ with its Borel σ-algebra $\mathcal{F}$. Outcomes are denoted by $\omega = (u,v)$ with $u \in [0,1], v \in \{-1,+1\}$, and we define $U(\omega) = u$ the first and by $V(\omega) = v$ the second coordinate. Let $\mathcal{F}_0 = \mathcal{F}_1 = \sigma(U)$, $\mathcal{F}_2 = \mathcal{F}$ and let $\mathbb{P}$ be the measure on $(\Omega, \mathcal{F})$ such that U is distributed uniformly on $[0,1]$ and the conditional distribution of V given U is $U^2\delta_{\{+1\}} + (1-U^2)\delta_{\{-1\}}$. Let $X_0 = 0, \Delta X_1 = 1$ and

$$\Delta X_2 = V^+(1+U) - 1 = V^+U - V^-,$$

so that

$$\Delta X_2(u,v) = u\delta_{\{v=+1\}} - \delta_{\{v=-1\}}$$

. Consider now the contingent claim $H = V(1+U)^+$.

i) Show that $H \in L^2(\Omega, \mathcal{F}, \mathbb{P})$
ii) Let ϕ be a predictable process with terminal gain satisfying $G_2(\phi) = H$ $\mathbb{P}$-almost surely. Show that

$$\frac{1}{U}V^+(1+U) = H = \phi_1\Delta X_1 + \phi_2\Delta X_2 = \phi_1 + \phi_2(V^+(1+U) - 1) \tag{3}$$

implies that $\phi_1 = \phi_2 = U^{-1}$ $\mathbb{P}$-almost surely.
iii) Show that ϕ is not in $\mathcal{S}^1$ and conclude that H is not in $\mathcal{G}_2$.
iv) Next, set

$$\phi^n := \phi\mathbb{1}_{\{U>1/n\}} = U^{-1}\mathbb{1}_{\{U>1/n\}} \tag{4}$$

and show that $\phi^n \in \mathcal{S}^1$ for every $n \in \mathbb{N}$
and that

$$\phi_2^n\Delta X_2 = U^{-1}V^+(1+U)\mathbb{1}_{\{U>1/n\}} = H\mathbb{1}_{\{U>1/n\}} \tag{5}$$

converges to H in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.
v) Part i)-iv) shows that the space $\mathcal{G}_2$ is not closed in $L^2(\Omega, \mathcal{F}, \mathbb{P})$, so the variance optimization problem for H does not have a solution. To conclude this example, show that X as constructed above does not satisfy the bounded mean-variance trade-off condition.